Lecture Note for Differential Geometry Course, fall, 2023

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1 Chapter 1: Curves.

1.1 Introduction.

See textbook p. 1.

1.2 Parametrized Curves (this is section 1-2 of the book).

There are several ways to introduce a "curve in $\mathbb{R}^3$". For example, one can view it as the intersection of two regular surfaces in$\mathbb{R}^3$, or view it as a subset satisfying the definition in p. 75 (similar to the definition of a regular surface in p. 52), or view it as the "trace (i.e., image)" of a differentiable map $\alpha : I \to \mathbb{R}^3$. However, from the viewpoint of differentiable calculus (since we want to "do calculus on the curve" and "to find some important quantities related to derivatives" !!!), the best way is to view it as a differentiable map, not just a subset of $\mathbb{R}^3$ (there is an ant *crawling* on the curve). Thus we give the following definition:

Definition 1.1 *A* ***parametrized differentiable curve*** *is a* ***differentiable*** *(or* ***smooth****) map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$. The image set $\alpha(I)$ is called the* ***trace*** *of α, which is the usual "curves" you see in $\mathbb{R}^3$.*

Note the following:

- From now on, for convenience, we shall call a "**parametrized differentiable curve**" simply a "**differentiable curve**" of just a "**curve**" if no confusion occurs.
- The interval (a, b) can be $(-\infty, \infty)$. If we want to consider a curve with endpoints, we replace $I = (a, b)$ by $I = [a, b]$, where both a, b are finite.
- Writing $\alpha(t) = (x(t), y(t), z(t))$ for $t \in I$, the word "**differentiable**" means that $x(t)$, $y(t)$ and $z(t)$ have derivatives of **all** orders (we say they are C^∞ functions).

Definition 1.2 *The vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the* ***tangent vector*** *(or velocity vector) of the curve α at $t \in I$.*

- One should carefully distinguish a parametrized curve, which is a map, from its trace (image), which is a subset of $\mathbb{R}^3$. Two different curves in $\mathbb{R}^3$ may have the same trace.
- (**Interesting.**) There is a **continuous map** $\alpha : [0, 1] \to \mathbb{R}^3$ whose image is **the whole closed cube** $[0, 1] \times [0, 1] \times [0, 1]$ in $\mathbb{R}^3$ (or there is a **continuous map** $\alpha : [0, 1] \to \mathbb{R}^2$ whose image is **the whole closed square** $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$) (quite counter-intuitive !!!). Such a map cannot be one-one. They are known as **space-filling curves**.

Example 1.3 *Do Example 1-5 in the book p. 2-4. For Example 2 in the book, if we let $x = t^3$, $y = t^2$, we see that the whole parametrized curve $\alpha(t) = (t^3, t^2)$, $t \in (-\infty, \infty)$, has the same trace as the graph $y = x^{2/3}$, $x \in (-\infty, \infty)$. The function $y = x^{2/3}$ is not differentiable at $x = 0$. Its graph is given in p. 3 of the book. Note that $\alpha'(0) = (0, 0)$. We say $\alpha(t)$ has a* ***singular point*** *at $t = 0$.*

Remark 1.4 *Note in particular that in Example 2, a differentiable curve may have "****corners****" of "****cusps****" in its trace. These points occur at those points with* ***zero*** *tangent vector.*

Remark 1.5 *It will be better to use the definition for* ***inner product*** *as*

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

then the properties 1-4 in the bottom of p. 4 are quite easy to verify. Then from calculus we know that it is equivalent to the geometric definition

$$u \cdot v = |u|\,|v| \cos\theta.$$

Exercise 1.6 *Do exercise 1, 2, 4, 5, in p. 5.*

Example 1.7 *(**This interesting example is to compare with Example 4 in p. 3.**) It is interesting to see that there is a differentiable curve $\beta(t) : (-\infty, \infty) \to \mathbb{R}^2$ on $\mathbb{R}^2$ with its* ***trace*** *the same as the trace of the curve $\alpha(t) = (t, |t|) \in \mathbb{R}^2,\ t \in (-1, 1)$, which has a* ***corner*** *at the point $(0, 0)$. Note that $\alpha(t)$ is not a differentiable curve. To construct $\beta(t)$, we first define $x(t)$ as*

$$x(t) = \begin{cases} e^{-1/t^2}, & t \in (0, \infty), \\ 0, & t = 0, \\ -e^{-1/t^2}, & t \in (-\infty, 0). \end{cases}$$

*One can see that $x(t)$ is a differentiable function on $(-\infty, \infty)$ with $x^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Now let $\beta(t) : (-\infty, \infty) \to \mathbb{R}^2$ be $\beta(t) = (x(t),\ |x(t)|)$. We see that $x(t) \in (-1, 1)$ for all $t \in (-\infty, \infty)$. Also the function $|x(t)| \in [0, 1)$ is also a differentiable (C^∞) function on $(-\infty, \infty)$. We have $\beta'(0) = (0, 0)$ (**vanishing tangent vector**), which explains why we have a* ***corner*** *at $(0, 0)$. Compare with Example 2 in p. 3.*

1.3 Regular Curves; Arc Length (this is section 1-3 of the book).

Definition 1.8 *Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $\alpha'(t) \neq 0$, then the line L containing the point $\alpha(t)$ and pointing in the direction of $\alpha'(t)$ is called the* ***tangent line*** *to α at t (or the* ***tangent line*** *to α at $\alpha(t)$). The parametric representation of L is*

$$L = \{\alpha(t) + \lambda \alpha'(t) : \lambda \in \mathbb{R}\}.$$

Remark 1.9 *For the study of the differential geometry of a curve α it is essential that there exists such a tangent line* ***at every point*** *of the curve. We say t (or $\alpha(t)$) is a* ***singular point*** *of α if $\alpha'(t) = 0$.*

Definition 1.10 *Let $\alpha : I \to \mathbb{R}^3$ be a curve. If $\alpha'(t) \neq 0$ for all $t \in I$, then we say α is a* ***regular curve.***

Remark 1.11 ***From now on, we shall consider only regular parametrized differentiable curves.***

Definition 1.12 *The arc length function $s(t)$, measured from the point t_0 (call it the* ***origin*** *of α), of a regular curve $\alpha : I \to \mathbb{R}^3$ is defined by*

$$s(t) := \int_{t_0}^{t} |\alpha'(t)|\, dt, \quad t_0 \in I, \tag{1}$$

where $|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$, $t \in I$. There is a geometric reason for such a definition. See ex. 8 in p. 10.

For the arc length function $s(t)$ we have

$$s'(t) = |\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} > 0, \quad \forall\, t \in I.$$

Hence $s(t)$ is a strictly increasing differentiable (C^∞) function of $t \in I$ and it has differentiable inverse function $t = t(s)$ for $s \in J$ ($J = s(I)$). If we use the arc length parameter $s \in J$ to **reparametrize** the regular curve α, we get the new parametrized curve

$$\beta(s) := \alpha(t(s)), \quad s \in J$$

and by the chain rule we have

$$|\beta'(s)| = |\alpha'(t(s))\, t'(s)| = t'(s)\, |\alpha'(t)| = \frac{1}{s'(t)} |\alpha'(t)| = \frac{1}{|\alpha'(t)|} |\alpha'(t)| = 1, \quad \forall\, s \in J. \tag{2}$$

Thus if a curve is parametrized by arc length parameter s, **its tangent vector is of unit length everywhere**. Conversely if $\alpha(t)$ is a regular curve with $|\alpha'(t)| = 1$ everywhere, then by (1) we have

$$s = \int_{t_0}^{t} 1 dt = t - t_0,$$

which means that $t = s + t_0$ is the arc length of α measured from the point t_0. Note that one can always use arc length parameter s to parametrize a **regular curve** α.

Remark 1.13 *From now on, to simplify our exposition, we shall restrict ourselves to curves parametrized by arc length parameter s. We shall see later that this restriction is not essential (as long as the curve is **regular**). In general, it is not necessary to mention the origin of the arc length s since most important geometric quantities are expressed only in terms of the derivatives of $\alpha(s)$.*

Definition 1.14 *Given $\alpha(s)$, $s \in (a, b)$, the curve*

$$\beta(s) = \alpha(-s), \quad s \in (-b, -a) \tag{3}$$

*is called a **change of orientation** of the curve α. It has the same trace as α, but is described in the opposite direction. Note that (3) implies $\beta'(s) = -\alpha'(-s)$, i.e. the direction of the tangent vector is reversed.*

Remark 1.15 *Another parametrization for a change of orientation is to let $\beta(s) = \alpha(a + b - s)$, $s \in (a, b)$.*

Remark 1.16 *In p. 7 of the textbook, correct "$\beta(-s) = \alpha(s)$" as "$\beta(s) = \alpha(-s)$", $s \in (-b, -a)$.*

Remark 1.17 *This is to explain the statement in p. 6: "Since $\alpha'(t) \neq 0$, the arc length s is a differentiable (which means has derivatives of all orders) function of t and $ds/dt = |\alpha'(t)|$.". This is because we have*

$$\frac{ds}{dt} = |\alpha'(t)| = \left[(x'(t))^2 + (y'(t))^2 + (y'(t))^2\right]^{1/2},$$

where $(x'(t))^2 + (y'(t))^2 + (y'(t))^2 > 0$, and so we can compute d^2s/dt^2, d^3s/dt^3, ..., etc. Therefore, s is a differentiable function of t.

Remark 1.18 ***Any regular curve** $\alpha(t) : I \to \mathbb{R}^3$ **can be parametrized by arc length parameter** s (unique up to the choice of **the origin** of the arc length). This is because we have $ds/dt = |\alpha'(t)| > 0$ for all $t \in I$, then by the Inverse Function Theorem, the correspondence $s \longleftrightarrow t$ has **smooth inverse**, i.e. the variable $t \in I$ can be expressed as a smooth function of $s \in J$ (some interval). By this we have $\alpha(t) = \alpha(t(s)) : s \in J \to \mathbb{R}^3$, which becomes a differentiable curve parametrized by arc length parameter s.*

Exercise 1.19 *Do exercise 1, 2, 4, 6, 8, 10 in p. 7.*

Exercise 1.20 *It is known that there is a **continuous map** $\alpha : [0, 1] \to \mathbb{R}^2$ whose image is **the whole closed square** $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Explain that it cannot happen if we assume $\alpha : I \to \mathbb{R}^2$ is a **regular parametrized differentiable curve.***

1.4 The Vector Product in $\mathbb{R}^3$ (this is section 1-4 of the book).

Definition 1.21 *Let $e = \{e_1, e_2, e_3\}$ and $f = \{f_1, f_2, f_3\}$ be two ordered bases of the vector space $\mathbb{R}^3$. We say they have the **same orientation** if the 3×3 matrix of change of basis (explain this) has **positive determinant**. We denote this relation by $e \sim f$.*

Remark 1.22 *Using the relation*

$$\det\left(A^{-1}\right) = \frac{1}{\det A}, \quad \det A^T = \det A, \quad \det(AB) = (\det A)(\det B)$$

*one can easily check that the relation $\sim$ is an **equivalence relation** (i.e., (1). $e \sim e$; (2). if $e \sim f$, then $f \sim e$; (3). if $e \sim f$ and $f \sim g$, then $e \sim g$) among all ordered bases of $\mathbb{R}^3$.*

Definition 1.23 *Since the determinant of a **change of basis** is either positive or negative, there are only two such classes. Each of the equivalence classes determined by the above relation is called an **orientation** of $\mathbb{R}^3$. Hence $\mathbb{R}^3$ has **two** orientations. If we fix one of them, the other one is called the **opposite orientation**. The orientation containing the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ will be called the **positive orientation**. An ordered basis $\{u, v, w\}$ is called **positive** if $\{u, v, w\} \sim \{(1,0,0), (0,1,0), (0,0,1)\}$ (otherwise we call it **negative**). Thus $\{u, v, w\}$ is positive if and only if*

$$\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} > 0,$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$.

Remark 1.24 *We have encountered the concept of orientation in daily life. For example, on $\mathbb{R}^2$ we have "**clockwise orientation**" or "**counterclockwise orientation**" in turning faucets or screws.*

In linear algebra, there is the following result:

Lemma 1.25 *If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, then there exists an **unique** vector $s \in \mathbb{R}^3$ such that*

$$T(w) = s \cdot w, \quad \forall\, w \in \mathbb{R}^3, \tag{4}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^3$. The result remains true if we replace $\mathbb{R}^3$ by $\mathbb{R}^n$.

By the above lemma, we can define the following:

Definition 1.26 *For fixed u and v in $\mathbb{R}^3$, the map $T : \mathbb{R}^3 \to \mathbb{R}$ given by*

$$T(w) = \det(u, v, w) \ \ (= \det(w, u, v)), \quad w \in \mathbb{R}^3 \tag{5}$$

*is a linear transformation. By Lemma 1.25, the unique vector $s \in \mathbb{R}^3$ in (4) is called the **vector product** (or **cross product**) of u and v and is denoted as $u \wedge v \in \mathbb{R}^3$. Therefore, we have*

$$\det(u, v, w) = (u \wedge v) \cdot w, \quad \forall\, w \in \mathbb{R}^3, \tag{6}$$

which is the same as

$$\det(w, u, v) = w \cdot (u \wedge v), \quad \forall\, w \in \mathbb{R}^3. \tag{7}$$

Remark 1.27 *In $\mathbb{R}^n$ with $n > 3$, there in no corresponding concept for the vector product of **two** vectors. Instead, we can define the vector product $u_1 \wedge \cdots \wedge u_{n-1} \in \mathbb{R}^n$ of $n-1$ vectors $u_1, \ldots, u_{n-1} \in \mathbb{R}^n$ by the identity*

$$\det(u_1, \ \ldots, \ u_{n-1}, \ w) = (u_1 \wedge \cdots \wedge \ u_{n-1}) \cdot w, \quad \forall\, w \in \mathbb{R}^n. \tag{8}$$

Lemma 1.28 *If we express* $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, *then from the definition we can obtain*

$$u \wedge v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3,$$

which can be formally written as (for convenience of memory)

$$u \wedge v = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

and they have the following properties:

1. $u \wedge v = -(v \wedge u)$.
2. $u \wedge v$ depends linearly on u and v. That is, for any $a,\ b \in \mathbb{R}$ and any $u,\ v,\ w \in \mathbb{R}^3$ we have

$$(au + bw) \wedge v = au \wedge v + bw \wedge v.$$

3. $u \wedge v = 0$ if and only if $u,\ v$ are linearly dependent.
4. $(u \wedge v) \cdot u = (u \wedge v) \cdot v = 0$, i.e. $u \wedge v$ is **perpendicular** to both u and v.
5. If $u \wedge v \neq 0$, then $\{u,\ v,\ u \wedge v\}$ is a **positive** basis due to

$$\det(u,\ v,\ u \wedge v) = (u \wedge v) \cdot (u \wedge v) = |u \wedge v|^2 > 0.$$

6. For any 4 vectors $u,\ v,\ x,\ y \in \mathbb{R}^3$, we have

$$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & u \cdot y \\ v \cdot x & v \cdot y \end{vmatrix}. \tag{9}$$

In particular, we have

$$|u \wedge v|^2 = \begin{vmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{vmatrix} = |u|^2 |v|^2 \left(1 - \cos^2 \theta\right) = A^2, \tag{10}$$

where $\theta \in [0, \pi]$ is the angle between u and v; and A is the area of the parallelogram generated by $u,\ v$.

Remark 1.29 ***(Remark on p. 14.)*** *It is better to write the top identity in p. 14 as*

$$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & u \cdot y \\ v \cdot x & v \cdot y \end{vmatrix} \left(\text{not} \begin{vmatrix} u \cdot x & v \cdot x \\ u \cdot y & v \cdot y \end{vmatrix} \right).$$

Then it is easier to remember. By this, we have

$$(u \wedge v) \cdot (x \wedge y) = \det\left(A^T B\right), \quad A = (u, v), \quad B = (x, y)$$

where $A = (u, v)$ *(view* $u,\ v$ *as column vectors),* $B = (x, y)$ *(view* $x,\ y$ *as column vectors). This will be consistent with the identities in p. 16 and 17. Remember that in the 1-dimensional case, we have* $u \cdot x = \det\left(u^T x\right)$ *(view* $u,\ x$ *as column matrices).*

7. For any 3 vectors $u,\ v,\ w \in \mathbb{R}^3$, we have

$$(u \wedge v) \wedge w = (u \cdot w)\, v - (v \cdot w)\, u. \tag{11}$$

In particular, we note that

$$(u \wedge v) \wedge w \text{ (lies on } uv\text{-plane)} \neq u \wedge (v \wedge w) \text{ (lies on } vw\text{-plane)}.$$

Therefore, the vector product is **not** associative. In $\mathbb{R}^3$, the notation $u \wedge v \wedge w$ can be **confusing** (however, in $\mathbb{R}^4$ the notation $u \wedge v \wedge w$ makes sense).

8. For any two differentiable maps $u(t) : (a,b) \to \mathbb{R}^3,\ v(t) : (a,b) \to \mathbb{R}^3$, we have

$$\frac{d}{dt}\left(u(t) \wedge v(t)\right) = \frac{du}{dt} \wedge v(t) + u(t) \wedge \frac{dv}{dt}, \quad t \in (a,b).$$

Proof. We only prove (11). Since $(u \wedge v) \wedge w$ is linear in $u,\ v,\ w$, it suffices to look at $(e_i \wedge e_j) \wedge e_k$, where $1 \le i,\ j,\ k \le 3$ (here $\{e_1, e_2, e_3\}$ is the standard basis in $\mathbb{R}^3$). If $i = j$, then (11) clearly holds. Therefore, we look at $i \neq j$ and can also assume that $i < j$ (otherwise we change e_i and e_j). We have the following 9 cases to verify:

$$\begin{cases} (e_1 \wedge e_2) \wedge e_1, \quad (e_1 \wedge e_2) \wedge e_2, \quad (e_1 \wedge e_2) \wedge e_3 \text{ (this is 0)} \\ (e_1 \wedge e_3) \wedge e_1, \quad (e_1 \wedge e_3) \wedge e_2 \text{ (this is 0)}, \quad (e_1 \wedge e_3) \wedge e_3 \\ (e_2 \wedge e_3) \wedge e_1 \text{ (this is 0)}, \quad (e_2 \wedge e_3) \wedge e_2, \quad (e_2 \wedge e_3) \wedge e_3. \end{cases}$$

You can verify them by yourself. □

Exercise 1.30 *Do exercise 2, 5, 8, 10, 11, 12, 13 in p. 15.*

1.5 The Local Theory of Curves Parametrized by Arc Length (this is section 1-5 of the book).

Remark 1.31 *From now on, the parameter s is reserved for **arc length parameter** unless otherwise stated.*

From now on, we only focus on a regular curve $\alpha : I \to \mathbb{R}^3$ with $\alpha'(t) \neq 0$ for all $t \in I$. We can always reparametrize it by arc length parameter s. In the following, we always assume $\alpha : I \to \mathbb{R}^3$ is a regular curve parametrized by arc length parameter $s \in I$. It satisfies $|\alpha'(s)| = 1$ for all $s \in I$.

Definition 1.32 *Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number*

$$k(s) := |\alpha''(s)| = \left|\frac{dt}{ds}(s)\right| = |t'(s)| \ge 0, \quad s \in I \tag{12}$$

*is called the **curvature** of α at s, where $t(s) = \alpha'(s)$ is the unit tangent vector of α at s.*

Remark 1.33 *By definition, the curvature $k(s)$ cannot be negative. $k(s_0)$ **measures the rate of change of the tangent vectors near** s **with the tangent vector at** s. See Section 1.5.1 below also.*

Lemma 1.34 *A regular curve $\alpha(s) : I \to \mathbb{R}^3$ has $k(s) \equiv 0$ everywhere if and only if it is a straight line (or part of it) in $\mathbb{R}^3$.*

Proof. If α is straight line, since $|\alpha'(s)| = 1$ everywhere, we must have $\alpha(s) = us + v$ for some constant vectors $u,\ v \in \mathbb{R}^3$ (with $|u| = 1$). Hence $k(s) = |\alpha''(s)| = 0$ everywhere. Conversely if $k(s) = |\alpha''(s)| \equiv 0$, then $\alpha''(s) = 0$ and so $\alpha'(s) = u$ for some unit constant vector. Hence $\alpha(s) = us + v$ and the curve is a straight line. $\square$

Remark 1.35 *If we let $\beta(s) = \alpha(-s)$, then β is a change of orientation of the curve α. One trivially has*

$$\beta'(s) = -\alpha'(-s), \quad \beta''(s) = \alpha''(-s).$$

Therefore $\alpha''(s)$ and the curvature $k(s)$ remain ***invariant*** *under a change of orientation. Draw a picture to explain this ....*

Definition 1.36 *At points where $k(s) \neq 0$ (i.e., $k(s) > 0$) a unit vector $n(s)$ in the direction $\alpha''(s)$ is well-defined by the equation*

$$\alpha''(s) = \frac{dt}{ds}(s) = k(s)\,n(s), \quad k(s) > 0,$$

i.e. we define

$$n(s) = \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{k(s)}, \quad |n(s)| = 1. \tag{13}$$

By the identity $\frac{d}{ds}(\alpha'(s) \cdot \alpha'(s)) = 0$, $n(s)$ is perpendicular to the unit tangent vector $t(s) = \alpha'(s)$. We call it the ***normal vector*** *of α at s. The plane determined by the two vectors $t(s)$ and $n(s)$ is called the* ***osculating plane*** *of α at s.*

Remark 1.37 *At points where $k(s) = 0$, the normal vector $n(s)$ is undefined (so is the osculating plane). If $k(s) = 0$ (same as $\alpha''(s) = 0$), we say α has a* ***zero curvature point*** *at s (the terminology is different from textbook; also note that we always have $\alpha'(s) \neq 0$ for all s because it is an unit vector). Since the* ***osculating plane*** *is essential for us to study a regular curve, we shall restrict ourselves to curves parametrized by arc length* ***without*** *zero curvature point, i.e. we focus only on curves $\alpha(s)$ with $k(s) > 0$* ***everywhere****.*

Remark 1.38 *If $\beta(s) = \alpha(-s)$ is a change of orientation of the curve α, then*

$$t_\beta(s) = -t_\alpha(-s), \qquad n_\beta(s) = n_\alpha(-s). \tag{14}$$

1.5.1 Geometric Meaning of the Curvature $k(s)$.

Note that for a regular curve we have $|\alpha'(s)| = 1$ for all s. **From now on, we also assume $k(s) > 0$ for all s unless otherwise stated.** This implies

$$\langle \alpha''(s), \alpha'(s) \rangle = 0, \quad \forall\ s$$

and then

$$\begin{aligned} 0 &= \frac{d}{ds}\langle \alpha''(s), \alpha'(s) \rangle \\ &= \langle \alpha'''(s), \alpha'(s) \rangle + \langle \alpha''(s), \alpha''(s) \rangle = \langle \alpha'''(s), \alpha'(s) \rangle + k^2(s), \quad \forall\ s. \end{aligned} \tag{15}$$

Now if we have two **nearby** tangent vectors $\alpha'(s_0) \in \mathbb{R}^3$ and $\alpha'(s_0 + h) \in \mathbb{R}^3$, $h \in (-\varepsilon, \varepsilon)$ small, their small angle $\theta(h) \geq 0$, $\theta(h) \in [0, \delta)$, is given by the identity

$$\langle \alpha'(s_0), \alpha'(s_0 + h) \rangle = \cos\theta(h), \quad \theta(h) = \cos^{-1}\langle \alpha'(s_0), \alpha'(s_0 + h) \rangle \geq 0, \qquad \theta(0) = 0. \tag{16}$$

Note that here the angle $\theta(h)$ between $\alpha'(s_0) \in \mathbb{R}^3$ and $\alpha'(s_0+h) \in \mathbb{R}^3$ is always taken to be **nonnegative**. **Therefore, the limit**

$$\theta'(0) = \lim_{h\to 0} \frac{\theta(h) - \theta(0)}{h} = \lim_{h\to 0} \frac{\theta(h) \ \text{(nonnegative)}}{h \ \text{(positive or negative)}} \tag{17}$$

does not exist in general (use a picture to convince yourself; see Lemma 1.41 below). By the above observation, we can only look at $\theta'(0+)$ or $\theta'(0-)$.

To go on, we need the following calculus result:

Lemma 1.39 *For constant $a > 0$, we have*

$$\lim_{h\to 0^+} \frac{\cos^{-1}(1-ah)}{h} \quad \textit{does not exist} \tag{18}$$

and

$$\lim_{h\to 0^+} \frac{\cos^{-1}(1-ah^2)}{h} = \sqrt{2a}, \qquad \lim_{h\to 0^-} \frac{\cos^{-1}(1-ah^2)}{h} = -\sqrt{2a} \tag{19}$$

and

$$\lim_{h\to 0^+} \frac{\cos^{-1}(1-ah^3)}{h} = 0. \tag{20}$$

Remark 1.40 *By (19) and (20), one can show that*

$$\lim_{h\to 0^+} \frac{\cos^{-1}(1-ah^2+O(h^3))}{h} = \sqrt{2a}, \tag{21}$$

where $O(h^3)$ is any quantity depending on h satisfying

$$\left| \frac{O(h^3)}{h^3} \right| \le M, \quad \forall\ h \in (0,\varepsilon) \tag{22}$$

for some constant $M > 0$ and some small constant $\varepsilon > 0$.

Proof. Exercise. □

Now we can state the following:

Lemma 1.41 *Let $\alpha(s)$ be a regular curve with $k(s) > 0$ for all s. The angle $\theta(h)$ defined in (16) satisfies*

$$\theta'(0+) = k(s_0), \quad \theta'(0) \ \textit{does not exist}, \quad \theta'(0-) = -k(s_0). \tag{23}$$

Remark 1.42 *The curvature $k(s_0) = \theta'(0+)$ **measures the rate of change of angles** $\theta(h)$ between two nearby unit tangent vectors $t(s_0) \in \mathbb{R}^3$ and $t(s_0+h) \in \mathbb{R}^3$, $h > 0$. Here the derivative $\theta'(0+)$ is with respect to the arc length parameter (note that h is arc length parameter).*

Proof. We prove the first identity only. For fixed s_0 and small $h \in (-\varepsilon, \varepsilon)$ we let

$$g(h) = \langle \alpha'(s_0), \alpha'(s_0+h) \rangle = \cos\theta(h), \quad h \in (-\varepsilon,\varepsilon), \quad g(h) \in (1-\delta, 1].$$

Note that $g(h)$ is a C^∞ function on $(-\varepsilon,\varepsilon)$ with $g(0) = 1$, $g'(0) = 0$, and by the Taylor Theorem and (15) we have

$$\begin{aligned} g(h) &= \underbrace{g(0) + g'(0)h + \frac{1}{2!}g''(0)h^2}_{} + \widehat{\left(\frac{1}{3!}g'''(0)h^3 + \cdots\right)} \\ &= \underbrace{1 + \frac{1}{2}\langle \alpha'(s_0), \alpha'''(s_0)\rangle h^2}_{} + \widehat{O(h^3)} \ \text{(big } O \text{ notation)} \\ &= 1 - \frac{1}{2}k^2(s_0)h^2 + O(h^3), \quad g(h) \in (1-\delta, 1]. \end{aligned}$$

Hence

$$\theta(h) = \cos^{-1} g(h) = \cos^{-1}\left(1 - \frac{1}{2}k^2(s_0)h^2 + O(h^3)\right) \geq 0, \quad \theta(0) = 0, \quad h \in (-\varepsilon, \varepsilon),$$

which gives

$$\frac{\theta(h) - \theta(0)}{h} = \frac{\cos^{-1}\left(1 - \frac{1}{2}k^2(s_0)h^2 + O(h^3)\right)}{h}, \quad h \neq 0 \in (-\varepsilon, \varepsilon), \quad \theta(0) = 0. \tag{24}$$

By (21), we have

$$\theta'(0+) = \lim_{h \to 0^+} \frac{\cos^{-1}\left(1 - \frac{1}{2}k^2(s_0)h^2 + O(h^3)\right)}{h} = k(s_0)$$

and similarly $\theta'(0-) = -k(s_0)$, and $\theta'(0)$ does not exist (since $k(s_0) > 0$). The proof is done. □

Remark 1.43 *In case $k(s_0) = 0$, then the limit*

$$\lim_{h \to 0} \frac{\cos^{-1}(1 + O(h^3))}{h} = 0.$$

In such a case, $\theta(h)$ is differentiable at $h = 0$ with $\theta'(0+) = \theta'(0-) = \theta'(0) = 0$.

Remark 1.44 *(**Another intuitive quick way to see Lemma 1.41.**) We let $\beta(s) = \alpha'(s)$. Then $\beta(s)$ is a **space curve lying on the sphere** S^2. It is a regular curve due to $\beta'(s) = \alpha''(s) \neq 0$ for all $s \in I$ (however, s may not be arc length parameter for $\beta(s)$). At fixed $s = s_0$, the three points $O = (0,0,0)$, $A = \alpha'(s_0)$ and $B = \alpha'(s_0 + h)$ lie on the same plane and the same **circle with radius** 1 and the angle between OA and OB is $\theta(h) \geq 0$. When $h \in (-\varepsilon, \varepsilon)$ is small, we have (imagine $A \in S^2$ is the north pole)*

$$\begin{aligned} \theta(h) &= \text{circle arc length joining } A,\ B, \qquad \theta(h) \geq 0 \\ &\approx \text{arc length of the curve } \beta \text{ between } A,\ B \\ &= \begin{cases} \int_{s_0}^{s_0+h} |\alpha''(s)|\, ds, & h \in (0, \varepsilon) \\ \int_{s_0+h}^{s_0} |\alpha''(s)|\, ds, & h \in (-\varepsilon, 0). \end{cases} \end{aligned} \tag{25}$$

Therefore, we have

$$\theta'(0+) = |\alpha''(s_0)| = k(s_0), \qquad \theta'(0-) = -|\alpha''(s_0)| = -k(s_0). \tag{26}$$

Remark 1.45 *If the curve $\alpha(s)$ is a **plane curve**, then there is a convention to give the curvature $k(s)$ a sign (called **signed curvature**); see p. 22 of the textbook. In such a case, the angle $\theta(s)$ can be **positive or negative (in plane we have clockwise and counterclockwise orientation, but in space we do not have)** and $\theta(s)$ is a **differentiable** function of s everywhere with $\theta'(s) = k(s)$ for all s. See p. 23, Exercise 3.*

1.5.2 Geometric Meaning of the Torsion $\tau(s)$.

Definition 1.46 *Let $\alpha(s)$ be a regular curve with $k(s) > 0$ for all s. We know that the normal vector $n(s)$ is **well-defined everywhere**. The unit vector*

$$b(s) := t(s) \wedge n(s), \quad |b(s)| = 1, \quad \langle b(s), b'(s) \rangle \equiv 0 \tag{27}$$

*is called the **binormal vector** of α at s. This vector $b(s)$ is always perpendicular to the **osculating plane** at s. The three vectors $\{t(s),\ n(s),\ b(s)\}$ forma a positive basis in $\mathbb{R}^3$.*

Remark 1.47 *If $\beta(s) = \alpha(-s)$ is a change of orientation of the curve α, then $b_\beta(s) = -b_\alpha(-s)$. This is due to (27).*

Compute

$$b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s) = t(s) \wedge n'(s), \qquad t'(s) = k(s)\, n(s)$$

and we know that the vector $b'(s)$ is perpendicular to both $b(s)$ and $t(s)$, and so it must be pointing in the direction of $n(s)$. We may write

$$b'(s) = \tau(s)\, n(s) \tag{28}$$

for some function $\tau(s)$. The quantity $|b'(s)|$ measures the rate of change of the osculating planes near s with the osculating plane at s.

Definition 1.48 *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with $k(s) > 0$ everywhere. The number $\tau(s)$ defined by $b'(s) = \tau(s)\, n(s)$ is called the* ***torsion*** *of α at s. Unlike the curvature $k(s)$, the torsion of a space curve $\alpha(s)$ can be* ***either positive or negative.*** *The sign of τ has a geometric meaning.*

Remark 1.49 ***(Important.)*** *The curvature $k(s)$ is a* ***second derivative*** *quantity. The torsion $\tau(s)$ is a* ***third derivative*** *quantity.*

Remark 1.50 *If $\beta(s) = \alpha(-s)$ is a change of orientation of the curve α, then $b_\beta(s) = -b_\alpha(-s)$ we get $b'_\beta(s) = b'_\alpha(-s)$, which implies that $\tau_\beta(s)\, n_\beta(s) = \tau_\alpha(-s)\, n_\alpha(-s)$. Since $n_\beta(s) = n_\alpha(-s)$, we have $\tau_\beta(s) = \tau_\alpha(-s)$.*

We can conclude the following:

Lemma 1.51 *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with $k(s) > 0$ everywhere. Then the curvature $k(s)$ and the torsion $\tau(s)$ are* ***invariant*** *under change of orientation.*

The quantity $\tau(s)$ can measure whether $\alpha(s)$ is a plane curve or not. We have:

Lemma 1.52 *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with $k(s) > 0$ everywhere. Then $\alpha(s)$ is a* ***plane curve*** *(i.e. $\alpha(I)$ is contained in a plane P) if and only if $\tau(s) \equiv 0$ for all $s \in I$.*

Remark 1.53 *The condition $k(s) > 0$ everywhere in the above lemma is* ***necessary.*** *If $k(s) = 0$ somewhere, it is possible to have a* ***non-planar*** *curve with $\tau(s) \equiv 0$ everywhere. See Exercise 10 in p. 26 of the textbook.*

Proof. ($\Longrightarrow$). This is clear since now both $t(s)$ and $n(s)$ lie on the plane P; hence $b(s)$ is a constant unit vector perpendicular to P.

($\Longleftarrow$). Since $k(s) > 0$ everywhere, the vector $b(s)$ is everywhere defined with $b'(s) \equiv 0$ everywhere (due to $\tau(s) \equiv 0$). Thus $b(s) = b_0$ is a constant for all s, and

$$\frac{d}{ds} \langle \alpha(s), b_0 \rangle = \langle t(s), b_0 \rangle = \langle t(s), b(s) \rangle = 0, \quad \forall\, s \in I.$$

Therefore, $\langle \alpha(s), b_0 \rangle$ must be a constant and $\alpha(s)$ is a plane curve with image contained in a plane perpendicular to b_0. □

Remark 1.54 ***(Important.)*** *Physically we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion). Hence we are led to the fact that $k(s)$ and $\tau(s)$ completely determine the* ***local behavior*** *(and also* ***global behavior****) of the curve. See the* ***Fundamental Theorem*** *in p. 20 of the book.*

Assume that $k(s) > 0$ everywhere so that $t(s),\ n(s),\ b(s)$ are defined everywhere. We first have

$$\langle b(s), b(s)\rangle = 1, \quad \langle b'(s), b(s)\rangle = 0, \quad \forall\ s$$

and then

$$0 = \frac{d}{ds}\langle b'(s), b(s)\rangle = \langle b''(s), b(s)\rangle + \langle b'(s), b'(s)\rangle = \langle b''(s), b(s)\rangle + \tau^2(s). \tag{29}$$

Now if we have two **nearby** binormal vectors $b(s_0) \in \mathbb{R}^3$ and $b(s_0+h) \in \mathbb{R}^3,\ h \in (-\varepsilon, \varepsilon)$ small, their small angle $\theta(h) \geq 0,\ \theta(h) \in [0, \delta)$, is given by the identity

$$\langle b(s_0), b(s_0+h)\rangle = \cos\theta(h), \quad \theta(h) = \cos^{-1}\langle b(s_0), b(s_0+h)\rangle \geq 0, \qquad \theta(0) = 0. \tag{30}$$

We have the following result similar to Lemma 1.41:

Lemma 1.55 *Let $\alpha(s)$ be a regular curve with $k(s) > 0$ for all s. The angle $\theta(h)$ defined in (30) satisfies*

$$\theta'(0+) = |\tau(s_0)|, \quad \theta'(0) \text{ does not exist}, \quad \theta'(0-) = -|\tau(s_0)|. \tag{31}$$

Proof. For fixed s_0 and small $h \in (-\varepsilon, \varepsilon)$ we let

$$g(h) = \langle b(s_0), b(s_0+h)\rangle = \cos\theta(h), \quad h \in (-\varepsilon, \varepsilon), \quad g(h) \in (1-\delta, 1].$$

Note that $g(h)$ is a C^∞ function on $(-\varepsilon, \varepsilon)$ with $g(0) = 1,\ g'(0) = 0$, and by the Taylor Theorem and (29) we have

$$\begin{aligned} g(h) &= \underbrace{g(0) + g'(0)h + \frac{1}{2!}g''(0)h^2}+\overbrace{\left(\frac{1}{3!}g'''(0)h^3 + \cdots\right)} \\ &= \underbrace{1 + \frac{1}{2}\langle b(s_0), b''(s_0)\rangle} h^2 + \widehat{O(h^3)} \text{ (big } O \text{ notation)} \\ &= 1 - \frac{1}{2}\tau^2(s_0)h^2 + O\left(h^3\right), \quad g(h) \in (1-\delta, 1]. \end{aligned}$$

Hence

$$\theta(h) = \cos^{-1} g(h) = \cos^{-1}\left(1 - \frac{1}{2}\tau^2(s_0)h^2 + O\left(h^3\right)\right) \geq 0, \quad \theta(0) = 0, \quad h \in (-\varepsilon, \varepsilon).$$

which gives

$$\frac{\theta(h) - \theta(0)}{h} = \frac{\cos^{-1}\left(1 - \frac{1}{2}\tau^2(s_0)h^2 + O(h^3)\right)}{h}, \quad h \in (-\varepsilon, \varepsilon), \quad \theta(0) = 0. \tag{32}$$

By Lemma 1.39, we have (one can ignore the smallest term $O(h^3)$)

$$\begin{aligned} \theta'(0+) &= \lim_{h\to 0^+}\frac{\cos^{-1}\left(1 - \frac{1}{2}\tau^2(s_0)h^2 + O(h^3)\right)}{h} \\ &= \lim_{h\to 0^+}\frac{\cos^{-1}\left(1 - \frac{1}{2}\tau^2(s_0)h^2\right)}{h} = \sqrt{2 \cdot \frac{1}{2}\tau^2(s_0)} = |\tau(s_0)|. \end{aligned}$$

and similarly $\theta'(0-) = -|\tau(s_0)|$, and $\theta'(0)$ does not exist if $\tau(s_0) \neq 0$. The proof is done. □

Remark 1.56 *Similar to (25) in Remark 1.44, for small $h \in (-\varepsilon, \varepsilon)$, we have*

$$\theta(h) \approx \begin{cases} \int_{s_0}^{s_0+h} |b'(s)|\, ds, & h \in (0, \varepsilon) \\ \int_{s_0+h}^{s_0} |b'(s)|\, ds, & h \in (-\varepsilon, 0) \end{cases}, \qquad \theta(h) \geq 0 \tag{33}$$

and so

$$\theta'(0+) = |b'(s_0)| = |\tau(s_0)| \quad \text{and} \quad \theta'(0-) = -|b'(s_0)| = -|\tau(s_0)|. \tag{34}$$

Thus $|\tau(s_0)| = \theta'(0+)$ ***measures the rate of change of angles $\theta(h)$ of two nearby binormal vectors $b(s_0)$ and $b(s_0+h)$ (or two nearby osculating planes of α), $h > 0$.***

1.5.3 Frenet Frame of a Space Curve.

Definition 1.57 *Let $\alpha(s)$ be a regular curve with $k(s) > 0$ for all s. The three orthonormal vectors $\{t(s), n(s), b(s)\}$ in $\mathbb{R}^3$ is referred to as the **Frenet trihedron** (or **Frenet frame**) of α at s. It has positive orientation (in the order of $t(s),\ n(s),\ b(s)$).*

Lemma 1.58 *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with $k(s) > 0$ everywhere. We have the following **Frenet frame equations** (or just **Frenet equations**)*

$$\begin{cases} t'(s) = k(s)\, n(s), \\ n'(s) = -k(s)\, t(s) - \tau(s)\, b(s), \\ b'(s) = \tau(s)\, n(s) \end{cases} \tag{35}$$

for all $s \in I$.

Remark 1.59 ***(Important observation.)*** *Since there is no other geometric quantities (other than $k(s)$ and $\tau(s)$) appearing on the right hand side of (35), the behavior of a space curve $\alpha(s)$ is completely determined by $k(s)$ and $\tau(s)$. See Theorem 1.72 below.*

Proof. It suffices to check the second equation

$$\begin{aligned} n'(s) &= (b \wedge t)'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) \\ &= [\tau(s)\, n(s)] \wedge t(s) + b(s) \wedge [k(s)\, n(s)] = -\tau(s)\, b(s) - k(s)\, t(s). \end{aligned}$$

The proof is done. □

Remark 1.60 *If a space curve α happens to lie on a plane, then there is no torsion $\tau(s)$ ($\tau(s) \equiv 0$ for all s) and $b(s)$ is a constant vector and the Frenet frame equation becomes*

$$\frac{d}{ds}\begin{pmatrix} t'(s) \\ n'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} t(s) \\ n(s) \end{pmatrix}, \quad b'(s) = 0. \tag{36}$$

It is more convenient to write the **Frenet frame equations** in the **matrix form** (just formally):

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}\begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \quad \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \in \mathbb{R}^9. \tag{37}$$

Note that the coefficient matrix $M(s)$ is **anti-symmetric,** i.e. $M^T(s) = -M(s)$ for all $s \in I$.

Lemma 1.61 *Let $\{v_1(t), v_2(t), ..., v_n(t)\}$, $t \in I$, be a time-dependent basis in $\mathbb{R}^n$ with $v_i'(t) = \sum_{j=1}^n a_{ij}(t)\, v_j(t)$, $1 \le i \le n$. We have:*

1. In $\{v_1(t), v_2(t), ..., v_n(t)\}$ is orthonormal for all $t \in I$, then the matrix $M(t) = (a_{ij}(t))_{n\times n} \in M(n)$ is anti-symmetric for all $t \in I$.

2. Let $M \in M(n)$ be anti-symmetric. We have $\langle v, Mv \rangle = 0$ for all $v \in \mathbb{R}^n$. Also, if n is odd, then $\det M = 0$.

3. Let $M(t) = (a_{ij}(t))_{n\times n} \in M(n)$ be anti-symmetric for all $t \in I$. Formally, we have

$$\left\langle \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{pmatrix}, \ M(t)\begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{pmatrix} \right\rangle = 0, \quad \forall\, t \in I \tag{38}$$

for all $v_1(t), v_2(t), ..., v_n(t) \in \mathbb{R}^n$, where the inner product in (38) means:

$$\left\langle v_1(t), \sum_{j=1}^{n} a_{1j}(t) v_j(t) \right\rangle + \left\langle v_2(t), \sum_{j=1}^{n} a_{2j}(t) v_j(t) \right\rangle + \cdots + \left\langle v_n(t), \sum_{j=1}^{n} a_{nj}(t) v_j(t) \right\rangle. \tag{39}$$

Proof. Exercise. □

The following is about the computation of $\tau(s)$ in terms of $\alpha(s)$:

Lemma 1.62 *In terms of $\alpha(s)$, the torsion $\tau(s)$ can be expressed as*

$$\tau(s) = -\frac{(\alpha'(s) \wedge \alpha''(s)) \cdot \alpha'''(s)}{k^2(s)} = -\frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{|\alpha''(s)|^2}. \tag{40}$$

Remark 1.63 *From (40) we see that the torsion $\tau(s)$ is a third derivative object, while the curvature $k(s)$ is a second derivative object. Also, from (40) we see that the torsion $\tau(s)$ is invariant under change of orientation.*

Proof. (This is a homework problem in the book.) We have

$$\begin{aligned} \alpha'''(s) &= \frac{d}{ds}(k(s) n(s)) = k'(s) n(s) + k(s)[-k(s) t(s) - \tau(s) b(s)] \\ &= \underbrace{-k^2(s) t(s) + k'(s) n(s) - k(s) \tau(s) b(s)}. \end{aligned} \tag{41}$$

Hence

$$\begin{aligned} &(\alpha'(s) \wedge \alpha''(s)) \cdot \alpha'''(s) \\ &= k(s)[t(s) \wedge n(s)] \cdot \left[-k^2(s) t(s) + k'(s) n(s) - k(s) \tau(s) b(s)\right] \\ &= k(s)[t(s) \wedge n(s)] \cdot [-k(s) \tau(s) b(s)] \\ &= k(s) b(s) \cdot [-k(s) \tau(s) b(s)] = -k^2(s) \tau(s). \end{aligned}$$

The proof is done. □

Remark 1.64 *By (41), we have*

$$\begin{cases} \langle \alpha'''(s), \alpha'(s) \rangle = -k^2(s), \quad \langle \alpha'''(s), \alpha''(s) \rangle = k(s) k'(s), \\ \langle \alpha'''(s), \alpha'(s) \wedge \alpha''(s) \rangle = -k^2(s) \tau(s) \end{cases} \tag{42}$$

and by (41), we have

$$|\alpha'''(s)| = \sqrt{k^4(s) + (k'(s))^2 + k^2(s) \tau^2(s)} \geq k(s) \sqrt{k^2(s) + \tau^2(s)}. \tag{43}$$

Remark 1.65 *(**Differentiability of curvature and torsion.**) Let*

$$J = \{s \in I : |\alpha''(s)| > 0\} \subset I.$$

Then $k(s)$ and $\tau(s)$ are differentiable functions on the open interval J (use (40) and $k(s) = |\alpha''(s)|$). In particular, if we assume $\alpha : I \to \mathbb{R}^3$ is a regular curve with $k(s) = |\alpha''(s)| > 0$ everywhere, then $k(s)$ and $\tau(s)$ are differentiable functions on the domain interval I.

Definition 1.66 *The tb plane is called* ***rectifying plane****. The nb plane is called* ***normal plane****. The line passing through $\alpha(s)$ containing $n(s)$ is called* ***principal normal*** *and the line passing through $\alpha(s)$ containing $b(s)$ is called* ***binormal****.*

Definition 1.67 *The number $R(s) = 1/k(s)$ is called the* ***radius of curvature*** *of α at s.*

Example 1.68 *A* ***circle*** *$\alpha(s)$ in $\mathbb{R}^3$ with radius $r > 0$ has* ***constant curvature*** *$k = 1/r$ everywhere. To see this, we first note that $\alpha(s)$ is a plane curve. Without loss of generality, we may assume it lies on the xy-plane parametrized by arc length parameter s by the following:*

$$\alpha(s) = \left(r\cos\frac{s}{r},\ r\sin\frac{s}{r}\right), \quad s \in (0, 2\pi r), \quad |\alpha'(s)| = 1.$$

Then we have $k(s) = |\alpha''(s)| = 1/r$ for all $s \in (0, 2\pi r)$.

The converse of the above example does not hold. One can find a space curve $\alpha(s) : I \to \mathbb{R}^3$ with **constant curvature** k but it **is not** a circle. See the **helix** example in Exercise 1, p. 23 (a helix has **constant curvature and constant torsion; the converse is also true !!!**). However,if $\alpha(s)$ is a plane curve, it is correct. We have:

Lemma 1.69 *(**Plane curves with constant curvature must be circles.**) Let $\alpha : I \to \mathbb{R}^3$ be a regular curve with $k(s) > 0$ everywhere. If $\alpha(s)$ is a* ***plane curve*** *(equivalent to $\tau(s) \equiv 0$ for all $s \in I$ by Lemma 1.52) and $k(s) = k > 0$ is a* ***constant*** *for all $s \in I$, then $\alpha(I)$ lies on a* ***circle*** *with radius $r = 1/k$.*

Proof. Without loss of generality, we assume $\alpha(s) = (x(s), y(s), 0)$ lies on the xy-plane. For convenience, we can just write it as $\alpha(s) = (x(s), y(s))$. We have $|\alpha'(s)|^2 = 1$ and $|\alpha''(s)|^2 = k$ (constant) for all s, which implies

$$\alpha'(s) \cdot \alpha''(s) = 0, \quad \alpha''(s) \cdot \alpha'''(s) = 0, \quad \forall\ s \in I,$$

where by (41) we have

$$\alpha'''(s) = -k^2 t(s) = -k^2 \alpha'(s),$$

and we conclude

$$\frac{d}{ds}\left[\alpha''(s) + k^2\alpha(s)\right] = 0, \quad \forall\ s \in I$$

and so there is a **constant vector** λ such that $\alpha''(s) + k^2\alpha(s) = \lambda$ for all $s \in I$. By a translation (replace $\alpha(s)$ by $\alpha(s) + p,\ p \in \mathbb{R}^3$), without loss of generality, we may assume $\lambda = 0$ and obtain the vector equation

$$\alpha''(s) + k^2\alpha(s) = 0, \quad \forall\ s \in I.$$

Now taking inner product of the above ODE with respect to $\alpha'(s)$ to get

$$0 = \left(\alpha''(s) + k^2\alpha(s)\right) \cdot \alpha'(s) = k^2\alpha(s) \cdot \alpha'(s) = \frac{k^2}{2}\frac{d}{ds}|\alpha(s)|^2, \quad \forall\ s \in I,$$

which gives

$$|\alpha(s)|^2 = C \text{ (some positive constant)}, \quad \forall\ s \in I.$$

Therefore, $\alpha(s)$ lies on the circle centered at $(0,0)$ with radius $r = 1/k$ (since $\alpha(s)$ has constant curvature $k > 0$). The proof is done. □

1.5.4 Variation of Frenet Equations.

Recall the Frenet equations, which are

$$\begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix} = \begin{pmatrix} k(s)\,n(s) \\ -k(s)\,t(s) - \tau(s)\,b(s) \\ \tau(s)\,n(s) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} := M(s) \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix} \tag{44}$$

where $\{t(s), n(s), b(s)\}$ is the Frenet frame along $\alpha(s),\ s \in I$ and $M(s)$ is an **anti-symmetric** 3×3 matrix with

$$\det M(s) = \det \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} = 0, \quad \forall\ s \in I$$

We can rewrite (44) as the following form, which we call it **Darboux equations:**

$$\begin{cases} t'(s) = [-\tau(s)\,t(s) + k(s)\,b(s)] \wedge t(s) \\ n'(s) = [-\tau(s)\,t(s) + k(s)\,b(s)] \wedge n(s) \\ b'(s) = [-\tau(s)\,t(s) + k(s)\,b(s)] \wedge b(s), \end{cases} \tag{45}$$

which can be **formally written as**

$$\mathbf{F}'(s) = \omega(s) \wedge \mathbf{F}(s), \quad \mathbf{F}'(s) = \begin{pmatrix} t'(s) \\ n'(s) \\ b'(s) \end{pmatrix}, \quad \omega(s) \wedge \mathbf{F}(s) = \begin{pmatrix} \omega(s) \wedge t(s) \\ \omega(s) \wedge n(s) \\ \omega(s) \wedge b(s) \end{pmatrix}. \tag{46}$$

The vector

$$\omega(s) = -\tau(s)\,t(s) + k(s)\,b(s) \in \mathbb{R}^3 \tag{47}$$

is called the **Darboux rotation vector** (or **angular velocity**) of α at s.

Remark 1.70 *It is actually quite easy to obtain the Darboux rotation vector given by (**??**). Assume the **existence** of Darboux rotation vector $\omega(s)$, which is required to satisfy the property*

$$t'(s) = \omega(s) \wedge t(s), \quad n'(s) = \omega(s) \wedge n(s), \quad b'(s) = \omega(s) \wedge b(s), \quad \forall\ s \in I. \tag{48}$$

Then it is easy to find its explicit form. We can express $\omega(s)$ as

$$\omega(s) = A(s)\,t(s) + B(s)\,n(s) + C(s)\,b(s), \quad s \in I$$

for some coefficient functions $A(s),\ B(s),\ C(s)$. Then by

$$\underbrace{k(s)\,n(s)} = t'(s) = \omega(s) \wedge t(s)$$
$$= [A(s)\,t(s) + B(s)\,n(s) + C(s)\,b(s)] \wedge t(s) = \underbrace{-B(s)\,b(s) + C(s)\,n(s)},$$

which gives $C(s) = k(s),\ B(s) = 0$, and so $\omega(s) = A(s)\,t(s) + k(s)\,b(s)$ for some $A(s)$. Then we look at

$$\underbrace{\tau(s)\,n(s)} = b'(s) = \omega(s) \wedge b(s)$$
$$= [A(s)\,t(s) + k(s)\,b(s)] \wedge b(s) = \underbrace{-A(s)\,n(s)},$$

which gives $A(s) = -\tau(s)$. *Therefore, we conclude*

$$\omega(s) = -\tau(s)\, t(s) + k(s)\, b(s).$$

Finally, we check that the above $\omega(s)$ *also satisfies the identity* $n'(s) = \omega(s) \wedge n(s)$. *We have*

$$n'(s) = \underbrace{-k(s)\, t(s) - \tau(s)\, b(s)}$$

and

$$\omega(s) \wedge n(s) = [-\tau(s)\, t(s) + k(s)\, b(s)] \wedge n(s) = \underbrace{-\tau(s)\, b(s) - k(s)\, t(s)},$$

as verified. Therefore, we conclude the existence of such a Darboux rotation vector $\omega(s)$ *satisfying (48) and it is given explicitly by*

$$\omega(s) = -\tau(s)\, t(s) + k(s)\, b(s), \quad s \in I. \tag{49}$$

Remark 1.71 *By (45), the Darboux rotation vector* $\omega(s)$ $(\omega(s) = -\tau(s)\, t(s) + k(s)\, b(s) \neq 0$ *due to* $k(s) > 0)$ *is* ***perpendicular*** *to* $t'(s)$, $n'(s)$ *and* $b'(s)$. *In particular, the three vectors* $\{t'(s), n'(s), b'(s)\}$ *are* ***linearly dependent****. They only* ***span a plane*** P *perpendicular to* $\omega(s)$. *This can be seen from the Frenet equation (44), which says that* $t'(s)$ *and* $b'(s)$ *are lying on the same line* L *(pointing in the direction of* $n(s)$*). Therefore, the plane* P *is spanned by* L *and* $n'(s)$.

1.5.5 Fundamental Theorem of the Local Theory of Curves in $\mathbb{R}^3$.

Theorem 1.72 *(**Fundamental theorem of space curves.**) Let* I *be an open interval and* k_0, $\tau_0 : I \to \mathbb{R}$ *be two differentiable functions with* $k_0(s) > 0$ *for each* $s \in I$*.Then there exists a regular curve* $\alpha : I \to \mathbb{R}^3$, *such that* $s \in I$ *is the* ***arc length parameter*** *of* α *and its curvature* k_α *and torsion* τ_α *are given by*

$$k_\alpha(s) = k_0(s) \quad \text{and} \quad \tau_\alpha(s) = \tau_0(s), \quad \forall\ s \in I. \tag{50}$$

Moreover, if $\bar{\alpha} : I \to \mathbb{R}^3$ *is another space curve (also parametrized by arc length) satisfying (50), then there exists a* ***rigid motion*** *(see the definition below)* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *such that* $\bar{\alpha} = T \circ \alpha$. *Note that a rigid motion* $T : \mathbb{R}^3 \to \mathbb{R}^3$ ***preserves*** *the orientation of* $\mathbb{R}^3$, *i.e.* $\det T = +1$.

Remark 1.73 *(**Important.**) The above theorem* ***fails*** *if we replace the condition* $k_0(s) > 0$ *for each* $s \in I$ *by* $k_0(s) \geq 0$ *for each* $s \in I$. *Use a simple picture to explain this. See Remark 1.90.*

Remark 1.74 *A* ***rigid motion*** $T : \mathbb{R}^3 \to \mathbb{R}^3$ *has the form*

$$Tx = Ax + b, \quad A \in O(3), \quad b \in \mathbb{R}^3, \tag{51}$$

where A *is a* 3×3 ***orthogonal matrix*** *(i.e.* $A^T A = I$ *or* $\langle Ax, Ay \rangle = \langle x, y \rangle$ *for all* $x,\ y \in \mathbb{R}^n$*) with* $\det A = 1$ *and* $b \in \mathbb{R}^3$ *is a constant vector. Note that an* ***orthogonal matrix*** A *has* $\det A = \pm 1$. *If* $\det A = +1$, *it preserves the orientation of an ordered basis; if* $\det A = -1$, *it reverses the orientation.*

We shall not give a complete proof of the theorem (need to use the existence and uniqueness result for a system of first order linear equation). However, a proof of the **uniqueness** (up to rigid motions) of curves having the same s, $k_0(s)$ and $\tau_0(s)$ is easy.

To prove it, we first observe the following:

Lemma 1.75 *If two space curves* $\alpha(s)$, $\bar{\alpha}(s)$, $s \in I$, *are related by* $\bar{\alpha}(s) = T(\alpha(s))$ *for all* $s \in I$, *where* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *is a* ***rigid motion****, then*

$$k_{\bar{\alpha}}(s) = k_\alpha(s) \quad \text{and} \quad \tau_{\bar{\alpha}}(s) = \tau_\alpha(s), \quad \forall\ s \in I. \tag{52}$$

Remark 1.76 ***(Important.)*** *If $T:\mathbb{R}^3\to\mathbb{R}^3$ has the form (51) but with* $\det A=-1$, *then we have*

$$k_{\bar\alpha}(s)=k_\alpha(s)\quad\text{and}\quad \tau_{\bar\alpha}(s)=-\tau_\alpha(s),\quad \forall\ s\in I. \tag{53}$$

Proof. (**Read it yourself.**) Let $Tx=Ax+b$, where A is **orthogonal** ($A^TA=I$) with $\det A=1$. Without loss of generality, we may assume $b=0$ since obviously a translation will not affect curvature and torsion. We have

$$\bar\alpha'(s)=A\alpha'(s)\quad\text{and}\quad\bar\alpha''(s)=A\alpha''(s)$$

and so

$$\begin{aligned}k_{\bar\alpha}^2(s)&=\langle\bar\alpha''(s),\bar\alpha''(s)\rangle=\langle A\alpha''(s),A\alpha''(s)\rangle\\&=\left\langle\alpha''(s),A^TA\alpha''(s)\right\rangle=\langle\alpha''(s),\alpha''(s)\rangle=k_\alpha^2(s),\end{aligned}$$

which gives $k_{\bar\alpha}(s)=k_\alpha(s)$. As for the torsion, by (40) we have

$$\tau_{\bar\alpha}(s)=-\frac{(\bar\alpha'(s)\wedge\bar\alpha''(s))\cdot\bar\alpha'''(s)}{|k_{\bar\alpha}(s)|^2}$$

where (note that $\det A=1$)

$$\begin{aligned}&(\bar\alpha'(s)\wedge\bar\alpha''(s))\cdot\bar\alpha'''(s)=(A\alpha'(s)\wedge A\alpha''(s))\cdot A\alpha'''(s)\\&=\det(A\alpha'(s),A\alpha''(s),A\alpha'''(s))=(\det A)(\det(\alpha'(s),\alpha''(s),\alpha'''(s)))\\&=\det(\alpha'(s),\alpha''(s),\alpha'''(s))=(\alpha'(s)\wedge\alpha''(s))\cdot\alpha'''(s),\end{aligned}$$

which gives $\tau_{\bar\alpha}(s)=\tau_\alpha(s)$. □

Proof of the uniqueness part of Theorem 1.72:

Proof. Assume there are two curves $\alpha=\alpha(s)$, $\bar\alpha=\bar\alpha(s)$ satisfying the conditions

$$k_\alpha(s)=k_{\bar\alpha}(s)=k_0(s)>0,\quad \tau_\alpha(s)=\tau_{\bar\alpha}(s)=\tau_0(s),\quad\forall\ s\in I.$$

Let $\{t(s_0),n(s_0),b(s_0)\}$ and $\{\bar t(s_0),\bar n(s_0),\bar b(s_0)\}$ be the Frenet frames of α and $\bar\alpha$ respectively at $s=s_0\in I$. There exists a rigid motion $T:\mathbb{R}^3\to\mathbb{R}^3$ taking $\alpha(s_0)$ to $\bar\alpha(s_0)$ and taking $\{t(s_0),n(s_0),b(s_0)\}$ to $\{\bar t(s_0),\bar n(s_0),\bar b(s_0)\}$. Thus, **after performing this rigid motion** T on α, we have (we still denote $T(\alpha)$ as α if no confusion occurs)

$$\alpha(s_0)=\bar\alpha(s_0),\quad (t(s_0),n(s_0),b(s_0))=\left(\bar t(s_0),\bar n(s_0),\bar b(s_0)\right) \tag{54}$$

and the Frenet frames $\{t(s),n(s),b(s)\}$ and $\{\bar t(s),\bar n(s),\bar b(s)\}$ of α and $\bar\alpha$ satisfy the equation (here we need to use the property from Lemma 1.75 that a rigid motion preserves curvature and torsion)

$$\begin{pmatrix}t'(s)\\n'(s)\\b'(s)\end{pmatrix}=M(s)\begin{pmatrix}t(s)\\n(s)\\b(s)\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\bar t'(s)\\\bar n'(s)\\\bar b'(s)\end{pmatrix}=M(s)\begin{pmatrix}\bar t(s)\\\bar n(s)\\\bar b(s)\end{pmatrix}, \tag{55}$$

where

$$M(s)=\begin{pmatrix}0&k_0(s)&0\\-k_0(s)&0&-\tau_0(s)\\0&\tau_0(s)&0\end{pmatrix},\quad M^T(s)=-M(s)\ \text{(anti-symmetric)}$$

and we have (54). In particular, we note that (here we use formal notation again)

$$\frac{1}{2}\frac{d}{ds}\left\{\left|t(s)-\bar{t}(s)\right|^2+\left|n(s)-\bar{n}(s)\right|^2+\left|b(s)-\bar{b}(s)\right|^2\right\}$$
$$=\left\langle\begin{pmatrix} t(s)-\bar{t}(s) \\ n(s)-\bar{n}(s) \\ b(s)-\bar{b}(s)\end{pmatrix}, \frac{d}{ds}\begin{pmatrix} t(s)-\bar{t}(s) \\ n(s)-\bar{n}(s) \\ b(s)-\bar{b}(s)\end{pmatrix}\right\rangle$$
$$=\left\langle\begin{pmatrix} t(s)-\bar{t}(s) \\ n(s)-\bar{n}(s) \\ b(s)-\bar{b}(s)\end{pmatrix}, M(s)\begin{pmatrix} t(s)-\bar{t}(s) \\ n(s)-\bar{n}(s) \\ b(s)-\bar{b}(s)\end{pmatrix}\right\rangle = 0, \quad \forall\, s\in I, \tag{56}$$

where the last identity is due to $M(s)$ is **anti-symmetric** (see (38) in Lemma 1.61). Hence we have

$$\left|t(s)-\bar{t}(s)\right|^2+\left|n(s)-\bar{n}(s)\right|^2+\left|b(s)-\bar{b}(s)\right|^2$$
$$\equiv \text{const.} = 0 \quad (\text{due to } (54)), \quad \forall\, s\in I,$$

i.e.

$$t(s)=\bar{t}(s), \quad n(s)=\bar{n}(s), \quad b(s)=\bar{b}(s), \quad \forall\, s\in I.$$

Since

$$\alpha'(s)=t(s)=\bar{t}(s)=\bar{\alpha}'(s), \quad s\in I$$

and $\alpha(s_0)=\bar{\alpha}(s_0)$, by integration, we must have $\alpha(s)=\bar{\alpha}(s)$ for all $s\in I$. Thus the two curves only differ by a rigid motion. □

1.5.6 Curvature and Torsion for Space Curves not Parametrized by Arc Length.

What happens if a regular curve $\alpha(t): I\to\mathbb{R}^3$ (or $\mathbb{R}^2$) is not parametrized by arc length parameter s? Then one can always reparametrize it by arc length parameter s and obtain a new parametrization $\beta(s)$. With $\beta(s)$ one can compute the curvature $k(t)$ and torsion $\tau(t)$ of $\alpha(t)$ at any point. Thus we define:

Definition 1.77 *The curvature $k_\alpha(t)$ of $\alpha(t)$ at t_0 is the curvature $k_\beta(s)$ of $\beta(s)$ at s_0, where $\beta(s_0)=\alpha(t_0)$. Also the torsion $\tau_\alpha(t)$ of $\alpha(t)$ at t_0 is the torsion $\tau_\beta(s)$ of $\beta(s)$ at s_0. That is: $k_\alpha(t)=k_\beta(s(t))$, $\tau_\alpha(t)=\tau_\beta(s(t))$., where $\beta(s(t))=\alpha(t)$.*

Remark 1.78 *The above definition is independent of the choice of the reparametrization $\beta(s)$.*

Let $\alpha(t): I\to\mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$, **not necessarily parametrized by are length**. Let $\beta(s)$ be a reparametrization of α by arc parameter $s=s(t)$ measured from $t_0\in I$. We require $\beta(s)$ to have the **same** orientation as $\alpha(t)$.Therefore $s(t)$ is a strictly increasing function of t and vice versa. Then we have $\beta(s)=\alpha(t(s))$ and

$$\beta'(s)=\alpha'(t(s))\,t'(s), \quad \underbrace{t'(s)=\frac{1}{|\alpha'(t(s))|}}, \quad s'(t)=|\alpha'(t)| \tag{57}$$

and

$$\beta''(s)=\alpha''(t(s))\,(t'(s))^2+\alpha'(t(s))\,t''(s), \tag{58}$$

where

$$t''(s)=\frac{-1}{|\alpha'(t)|^2}\frac{d}{ds}(|\alpha'(t)|)=\frac{-1}{|\alpha'(t)|^2}\frac{\left\langle\alpha'(t),\ \frac{d}{ds}(\alpha'(t))\right\rangle}{|\alpha'(t)|}=\frac{-1}{|\alpha'(t)|^2}\frac{\left\langle\alpha'(t),\ \alpha''(t)\,t'(s)\right\rangle}{|\alpha'(t)|}$$
$$=\underbrace{\frac{-\left\langle\alpha'(t),\ \alpha''(t)\right\rangle}{|\alpha'(t)|^4}}, \quad t=t(s) \tag{59}$$

and so

$$\beta''(s)=\frac{\alpha''(t(s))}{|\alpha'(t(s))|^2}-\frac{\langle\alpha'(t),\ \alpha''(t)\rangle}{|\alpha'(t)|^4}\alpha'(t(s)) \tag{60}$$

Hence the space curve $k_\beta(s)\geq 0$ satisfies

$$\begin{aligned}k_\beta^2(s)&=\langle\beta''(s),\ \beta''(s)\rangle\\&=\left\langle\frac{\alpha''(t)}{|\alpha'(t)|^2}-\frac{\langle\alpha'(t),\ \alpha''(t)\rangle}{|\alpha'(t)|^4}\alpha'(t),\ \frac{\alpha''(t)}{|\alpha'(t)|^2}-\frac{\langle\alpha'(t),\ \alpha''(t)\rangle}{|\alpha'(t)|^4}\alpha'(t)\right\rangle\\&=\frac{|\alpha''(t)|^2}{|\alpha'(t)|^4}-\frac{\langle\alpha'(t),\ \alpha''(t)\rangle^2}{|\alpha'(t)|^6}=\frac{|\alpha'(t)|^2|\alpha''(t)|^2-\langle\alpha'(t),\ \alpha''(t)\rangle^2}{|\alpha'(t)|^6}\end{aligned}$$

and by the identity $|v\wedge w|^2=|v|^2|w|^2-(v\cdot w)^2$, we derive the **curvature formula**

$$\underbrace{0\leq k_\alpha(t)=k_\beta(s)=\frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}.} \tag{61}$$

Remark 1.79 *In particular, if $\alpha(t)=(x(t),y(t),0):I\to\mathbb{R}^2\subset\mathbb{R}^3$ is a plane curve, then we have*

$$k_\alpha(t)=\frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}=\frac{|x'(t)y''(t)-y'(t)x''(t)|}{\left((x'(t))^2+(y'(t))^2\right)^{3/2}},\quad t\in I. \tag{62}$$

Compare with (82) below.

As for the torsion, we know that (see (40))

$$\tau_\beta(s)=-\frac{(\beta'(s)\wedge\beta''(s))\cdot\beta'''(s)}{|\beta''(s)|^2}=-(\beta'(s)\wedge\beta''(s))\cdot\frac{\beta'''(s)}{k_\beta^2(s)},$$

where by (57) and (58), we get

$$\begin{aligned}&\beta'(s)\wedge\beta''(s)\\&=[\alpha'(t)t'(s)]\wedge\left[\alpha''(t)(t'(s))^2+\alpha'(t)t''(s)\right]=\frac{\alpha'(t)\wedge\alpha''(t)}{|\alpha'(t)|^3}\end{aligned}$$

and so

$$\tau_\beta(s)=-\frac{\alpha'(t)\wedge\alpha''(t)}{|\alpha'(t)|^3}\cdot\frac{\beta'''(s)}{k_\beta^2(s)}.$$

Now note that

$$\beta'''(s)=\frac{d}{ds}\left[\alpha''(t)(t'(s))^2+\alpha'(t)t''(s)\right]=\frac{\alpha'''(t)}{|\alpha'(t)|^3}+A\alpha'(t)+B\alpha''(t)$$

for some coefficients $A,\ B$. Hence in terms of t, we get

$$\begin{aligned}\tau(t)&=-\frac{\alpha'(t)\wedge\alpha''(t)}{|\alpha'(t)|^3}\cdot\frac{1}{k_\beta^2(s)}\left(\frac{\alpha'''(t)}{|\alpha'(t)|^3}+A\alpha'(t)+B\alpha''(t)\right)\\&=-\frac{(\alpha'(t)\wedge\alpha''(t))\cdot\alpha'''(t)}{|\alpha'(t)|^6k_\beta^2(s)},\qquad k_\beta(s)=\frac{|\alpha'(t)\wedge\alpha''(t)|}{|\alpha'(t)|^3}>0\\&=\underbrace{-\frac{(\alpha'(t)\wedge\alpha''(t))\cdot\alpha'''(t)}{|\alpha'(t)\wedge\alpha''(t)|^2}}\quad\text{(assume the denominator is not zero),}\end{aligned} \tag{63}$$

which is the formula for the torsion in terms of t.

1.5.7 The Convention of a Plane Curve; Signed Curvature.

Recall that if $\alpha : I \to \mathbb{R}^3$ is a regular curve parametrized by arc length s, its curvature $k(s)$ is defined as $k(s) = |\alpha''(s)| \geq 0$. However, in the particular case of a plane regular curve $\alpha : I \to \mathbb{R}^2$, it is possible to give the curvature $k(s)$ a **sign** (call it **signed curvature**). For this purpose, let $\{e_1, e_2\} = \{(1,0), (0,1)\}$ be the natural basis of $\mathbb{R}^2$ and define the normal vector $n(s)$, $s \in I$, by requiring the basis $\{t(s), n(s)\}$ to have the **same orientation** as the basis $\{e_1, e_2\}$. Therefore, the normal vector $n(s)$ is **always** chosen as

$$t(s) = (x'(s), y'(s)), \quad n(s) = (-y'(s), x'(s)), \quad s \in I \tag{64}$$

This normal vector $n(s)$ **is uniquely defined everywhere along the curve** $\alpha(s)$ **even if** $t'(s) = \alpha''(s) = 0$ **somewhere** (we can not achieve this if $\alpha(s)$ is a space curve). With this, the curvature $k(s)$ is then **defined via the identity**

$$t'(s) = k(s)\,n(s) \;\; (\text{same as } k(s) = \langle t'(s), n(s)\rangle = \langle \alpha''(s), n(s)\rangle\,), \; t(s) = \alpha'(s), \; t'(s) = \alpha''(s), \tag{65}$$

where now $k(s)$ may be either positive or negative. **It is clear that** $|k(s)|$ **agrees with the previously defined curvature** $k(s)$ **for space curves. Also note that both** $k(s)$ **and** $n(s)$ **change sign when we change the orientation of** α (draw a picture to see this !!) or **change the orientation of** $\mathbb{R}^2$.

Remark 1.80 *Draw a picture ....*

Remark 1.81 ***(Important.)*** *From now on, for a plane curve α, its curvature $k(s)$ always means* ***signed curvature*** *unless otherwise stated. Also, when we say that $\{t(s), n(s)\}$ is a Frenet frame of a plane curve α, we always mean that it is under the above sense, i.e.,* $\det(t(s), n(s)) > 0$ *everywhere.*

For a plane curve, one can express the unit vector $t(s)$ as

$$t(s) = (\cos\theta(s), \sin\theta(s)), \quad s \in I. \tag{66}$$

where $\theta(s)$ is the angle from e_1 to $t(s)$ in the orientation of $\mathbb{R}^2$. Then, under the above convention, we have

$$n(s) = (-\sin\theta(s), \cos\theta(s)), \quad \det(t(s), n(s)) > 0 \tag{67}$$

and by the curvature definition identity $t'(s) = k(s)\,n(s)$, we have the formula for a plane curve:

$$k(s) = \frac{d\theta}{ds}(s) \quad \text{for all} \quad s \in I. \tag{68}$$

The Frenet frame equations for plane curves become the following

$$\frac{d}{ds}\begin{pmatrix} t(s) \\ n(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix}\begin{pmatrix} t(s) \\ n(s) \end{pmatrix}, \quad k(s) = \text{signed curvature}, \quad s \in I. \tag{69}$$

To see (69), we note that

$$0 = \frac{d}{ds}\langle t(s),\ n(s)\rangle = \langle t'(s),\ n(s)\rangle + \langle t(s),\ n'(s)\rangle = k(s) + \langle t(s),\ n'(s)\rangle$$

and since $\langle n(s),\ n'(s)\rangle = 0$, we must have $n'(s) = -k(s)\,t(s)$.

Remark 1.82 ***(important)*** *Note that (69) still holds if we use the previous definition of* $k(s) = |\alpha''(s)| \geq 0$.

Lemma 1.83 *Let $\alpha(s) : I \to \mathbb{R}^2$ be a plane curve. Then*

$$k(s) = \langle t'(s), n(s)\rangle = \langle \alpha''(s), n(s)\rangle = \det(\alpha'(s), \alpha''(s)), \quad s \in I. \tag{70}$$

Thus the sign of $k(s)$ informs us about the orientation of the basis formed by the velocity vector $\alpha'(s)$ and the acceleration vector $\alpha''(s)$ of the curve.

Proof. The last identity is due to

$$\det(\alpha'(s), \alpha''(s)) = \det(t(s), k(s)n(s)) = k(s)\det(t(s), n(s)) = k(s).$$

□

Remark 1.84 *(**Comparison.**) Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular curve given by $\alpha(s) = (x(s), y(s))$, $s \in I$. Its **signed curvature** $k_{sign}(s) \in (-\infty, \infty)$ is given by*

$$k_{sign}(s) = \det(\alpha'(s), \alpha''(s)) = \underbrace{x'(s)y''(s) - y'(s)x''(s)}\ \textit{(may be pos. or neg.)}, \quad s \in I. \tag{71}$$

*On the other hand, if we view $\alpha(s)$ as a space curve, its **nonnegative curvature** $k(s) \in [0, \infty)$ is given by*

$$k(s) = |\alpha''(s)| = \underbrace{\sqrt{(x''(s))^2 + (y''(s))^2}}, \quad s \in I. \tag{72}$$

One can check that (use the identities $\langle \alpha'(s), \alpha'(s)\rangle = 1$ and $\langle \alpha'(s), \alpha''(s)\rangle = 0$ for all s)

$$[x'(s)y''(s) - y'(s)x''(s)]^2 = (x''(s))^2 + (y''(s))^2, \quad \forall\ s \in I. \tag{73}$$

Therefore, we have

$$|x'(s)y''(s) - y'(s)x''(s)| = |k_{sign}(s)| = k(s) = \sqrt{(x''(s))^2 + (y''(s))^2}, \quad \forall\ s \in I. \tag{74}$$

Example 1.85 *Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular curve and lies on a a circle with radius $r > 0$. As s in increasing, if $\alpha(s)$ goes in the counterclockwise direction, then $k(s) = 1/r$ everywhere; if $\alpha(s)$ goes in the clockwise direction, then $k(s) = -1/r$ everywhere.*

There is another version of the fundamental theorem for plane curves where now the curvature $k_0(s)$ is **signed curvature**. We first note the following:

Lemma 1.86 *If two plane curves $\alpha(s)$, $\bar{\alpha}(s)$, $s \in I$, are related by $\bar{\alpha}(s) = T(\alpha(s))$ for all $s \in I$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a **rigid** motion (it is **orientation-preserving**), then they have the same signed curvature. That is*

$$k_\alpha(s) = k_{\bar{\alpha}}(s), \quad \forall\ s \in I. \tag{75}$$

Proof. Exercise. □

Theorem 1.87 *(**Fundamental theorem of plane curves.**). Let $k_0 : I \to \mathbb{R}$ be a differentiable function (not necessarily positive) defined on an open interval $I \subset \mathbb{R}$. Then, there exists a plane curve $\alpha : I \to \mathbb{R}^2$, **parametrized by arc length,** such that*

$$k_\alpha(s) = k_0(s) \quad \textit{for all} \quad s \in I$$

*where $k_\alpha(s) \in (-\infty, \infty)$ is the **signed curvature** function of α. Moreover, if $\bar{\alpha} : I \to \mathbb{R}^2$ is another plane curve (also parametrized by arc length) with*

$$k_{\bar{\alpha}}(s) = k_0(s) \quad \textit{for all} \quad s \in I$$

*then there exists a **rigid motion** (see the definition in Remark 1.74) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $\det T = 1$ such that $\bar{\alpha} = T \circ \alpha$.*

Remark 1.88 *A rigid motion $T:\mathbb{R}^2\to\mathbb{R}^2$ is either a rotation or a translation or a composition of both.*

Proof. Pick any three $s_0,\ s_1,\ s_2\in I$ and define

$$\theta(s)=\int_{s_0}^{s}k_0(u)\,du,\quad \alpha(s)=\left(\int_{s_1}^{s}\cos\theta(u)\,du,\ \int_{s_2}^{s}\sin\theta(u)\,du\right),\qquad s\in I.$$

$\alpha(s):I\to\mathbb{R}^2$ is clearly smooth with $|\alpha'(s)|=1$ everywhere. Hence $\alpha(s)$ is parametrized by arc length, with

$$\alpha'(s)=(\cos\theta(s),\sin\theta(s)),\qquad \alpha''(s)=\theta'(s)(-\sin\theta(s),\cos\theta(s))$$

and so

$$k_\alpha(s)=\det(\alpha'(s),\alpha''(s))=\theta'(s)=k_0(s)\quad\text{for all}\quad s\in I.$$

The claims the **existence** part.

To prove the **uniqueness**, suppose there are two plane curves $\alpha(s),\ \bar{\alpha}(s)$ satisfying the conditions

$$k_{\bar{\alpha}}(s)=k_\alpha(s)=k_0(s),\quad \forall\ s\in I.$$

Let $\{t_0,n_0\}$ and $\{\bar{t}_0,\bar{n}_0\}$ be the Frenet frames of α and $\bar{\alpha}$ respectively at $s=s_0\in I$. There exists a rigid motion $T:\mathbb{R}^2\to\mathbb{R}^2$ taking $\bar{\alpha}(s_0)$ to $\alpha(s_0)$ and $\{\bar{t}_0,\bar{n}_0\}$ to $\{t_0,n_0\}$. Thus, after performing this rigid motion on $\bar{\alpha}$, we have $\bar{\alpha}(s_0)=\alpha(s_0)$ and the Frenet frames $\{t(s),n(s)\}$ and $\{\bar{t}(s),\bar{n}(s)\}$ of α and $\bar{\alpha}$ satisfy the equation (here we need to use the above lemma)

$$\begin{pmatrix} t'(s)\\ n'(s)\end{pmatrix}=\begin{pmatrix} k_0(s)\,n(s)\\ -k_0(s)\,t(s)\end{pmatrix}\quad\text{and}\quad \begin{pmatrix} \bar{t}'(s)\\ \bar{n}'(s)\end{pmatrix}=\begin{pmatrix} k_0(s)\,\bar{n}(s)\\ -k_0(s)\,\bar{t}(s)\end{pmatrix}$$

with $t(s_0)=\bar{t}(s_0),\ n(s_0)=\bar{n}(s_0)$. In particular, we have (check it yourself)

$$\frac{1}{2}\frac{d}{ds}\left(|t(s)-\bar{t}(s)|^2+|n(s)-\bar{n}(s)|^2\right)=0,\quad \forall\ s\in I \tag{76}$$

and so $t(s)=\bar{t}(s),\ n(s)=\bar{n}(s)$ for all $s\in I$. Since

$$\alpha'(s)=t(s)=\bar{t}(s)=\bar{\alpha}'(s),\quad s\in I$$

we obtain $\alpha(s)=\bar{\alpha}(s)$ for all $s\in I$ (note that $\alpha(s_0)=\bar{\alpha}(s_0)$). Thus the two curves only differ by a rigid motion. □

Remark 1.89 *(**Important.**) The above proof is **constructive** in the sense that a plane curve $\alpha(s)$ parametrized by arc length length $s\in I$ whose signed curvature equal to a given function $k(s)$ can be found by solving the system of ODE:*

$$\begin{cases}\theta'(s)=k(s)\\ \alpha'(s)=(\cos\theta(s),\ \sin\theta(s))\end{cases},\qquad s\in I \tag{77}$$

and get $\alpha(s)$. The solution $\alpha(s)$ is unique up to a rigid motion $T:\mathbb{R}^2\to\mathbb{R}^2$.

Remark 1.90 *(**Important.**) The above theorem **fails** if the curvature is **not** signed curvature but given by $k(s)=|\alpha''(s)|\geq 0$. For example, consider the graph*

$$y=f(x),\quad x\in(-1,1).$$

*The (**nonnegative**) curvature is given by (using Exercise 12 in p. 26 of the book)*

$$k(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}, \quad x \in (-1,1).$$

Hence if we have $y = x^3,\ x \in (-1,1)$, *then*

$$k(x) = \frac{6|x|}{[1+9x^4]^{3/2}}, \quad x \in (-1,1).$$

On the other hand, if we have

$$y = \begin{cases} x^3, & x \in [0,1) \\ -x^3, & x \in (-1,0) \end{cases} \tag{78}$$

*then we have the **same** curvature $k(x)$ for all $x \in (-1,1)$. However, we cannot find a rigid motion to carry a curve into another. **One can also use this example to explain why we need $k > 0$ everywhere in Theorem 1.72. Note:** The curve given by (78) is not C^∞, but it does not matter too much. One can find another example which is a C^∞ curve.*

1.5.8 Some expressions for signed curvature $k(s)$ of plane curves parametrized by arc length.

Lemma 1.91 *If a plane curve $\alpha(s) = (x(s), y(s))$ is parametrized by arc length parameter $s \in I$, then we know that*

$$k(s) = x'(s)\,y''(s) - x''(s)\,y'(s) \tag{79}$$

and

$$k^2(s) = (x''(s))^2 + (y''(s))^2 = -\left[x'(s)\,x'''(s) + y'(s)\,y'''(s)\right]. \tag{80}$$

Moreover, we also have

$$k^3(s) = x''(s)\,y'''(s) - x'''(s)\,y''(s). \tag{81}$$

Proof. By $d\mathbf{t}/ds = k\mathbf{n},\ d\mathbf{n}/ds = -k\mathbf{t}$, we have (note that $\mathbf{t} = (x'(s), y'(s)),\ \mathbf{n} = (-y'(s), x'(s))$)

$$(x'''(s), y'''(s)) = \frac{d}{ds}(x''(s), y''(s)) = \frac{d}{ds}(k(s)\,\mathbf{n}(s)) = k_s(s)\,\mathbf{n}(s) - k^2(s)\,\mathbf{t}(s)$$

and noting that the vector $(-y''(s), x''(s)) = d\mathbf{n}/ds = -k\mathbf{t}$ is **tangential**, we obtain

$$\langle (x'''(s), y'''(s)),\ (-y''(s), x''(s)) \rangle = \left\langle k_s(s)\,\mathbf{n}(s) - k^2(s)\,\mathbf{t}(s),\ -k\mathbf{t} \right\rangle = k^3(s).$$

The proof is done. □

Remark 1.92 *One can also use the following more straightforward method:*

$$\begin{aligned} k^3(s) &= k^2(s) \cdot k(s) = -\left[x'(s)\,x'''(s) + y'(s)\,y'''(s)\right]\left[x'(s)\,y''(s) - x''(s)\,y'(s)\right] \\ &= \begin{cases} -(x'(s))^2\,x'''(s)\,y''(s) + \underline{x'(s)\,x''(s) \cdot y'(s)\,x'''(s)} \\ -\underbrace{y'(s)\,y''(s) \cdot x'(s)\,y'''(s)} + (y'(s))^2\,x''(s)\,y'''(s) \end{cases} \end{aligned}$$

and by

$$x'(s)\,x''(s) + y'(s)\,y''(s) = 0, \quad (x'(s))^2 + (y'(s))^2 = 1$$

we get

$$k^3(s) = \begin{cases} -(x'(s))^2 x'''(s)\, y''(s) - \underline{(y'(s))^2 y''(s)\, x'''(s)} \\ +\underbrace{(x'(s))^2 x''(s)\, y'''(s)} + (y'(s))^2 x''(s)\, y'''(s) \end{cases}$$
$$= -\left[(x'(s))^2 + (y'(s))^2\right] x'''(s)\, y''(s) + \left[(x'(s))^2 + (y'(s))^2\right] x''(s)\, y'''(s)$$
$$= x''(s)\, y'''(s) - x'''(s)\, y''(s).$$

The proof is done.

Remark 1.93 ***(elegant)*** *There is an elegant method. We know*

$$(x''(s), y''(s)) = k(s)(-y'(s), x'(s)) \qquad (\text{due to } \alpha''(s) = k(s)\, \mathbf{n}(s)).$$

Hence $x''(s) = -k(s)\, y'(s)$, $y''(s) = k(s)\, x'(s)$ *and then*

$$x''(s)\, y'''(s) - x'''(s)\, y''(s) = -k(s)\left[y'(s)\, y'''(s) + x'(s)\, x'''(s)\right] = k^3(s).$$

1.5.9 Signed Curvature for Plane Curves not Parametrized by Arc Length.

For regular plane curve $\alpha(t) = (x(t), y(t)) \in \mathbb{R}^2$ not parametrized by arc length, its **signed curvature** $k(t)$ is given by (this is almost obvious from (62))

$$k(t) = \frac{x'(t)\, y''(t) - y'(t)\, x''(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}. \tag{82}$$

To see this, first note that $\beta(s) = \alpha(t(s))$ and

$$\beta'(s) = \frac{\alpha'(t)}{|\alpha'(t)|} = \left(\frac{x'}{\sqrt{(x')^2 + (y')^2}}, \frac{y'}{\sqrt{(x')^2 + (y')^2}}\right) = (\cos\theta(s), \sin\theta(s))$$

due to (66). Next we recall that (see (60))

$$\beta''(s) = \underbrace{\frac{d\theta}{ds}}(-\sin\theta(s), \cos\theta(s)), \qquad k(s) = \frac{d\theta}{ds}$$
$$= \frac{\alpha''(t)}{|\alpha'(t)|^2} - \frac{\langle \alpha'(t),\ \alpha''(t)\rangle}{|\alpha'(t)|^4}\alpha'(t)$$
$$= \left(\frac{x''}{(x')^2 + (y')^2}, \frac{y''}{(x')^2 + (y')^2}\right) - \frac{x'x'' + y'y''}{\left((x')^2 + (y')^2\right)^2}(x',\ y')$$
$$= \underbrace{\frac{x'y'' - y'x''}{\left((x')^2 + (y')^2\right)^{3/2}}}\left(\frac{-y'}{\sqrt{(x')^2 + (y')^2}}, \frac{x'}{\sqrt{(x')^2 + (y')^2}}\right) = \underbrace{k(t)}(-\sin\theta(s), \cos\theta(s)),$$

which verifies (82).

Remark 1.94 *Another quick proof of (82) is: By* $k = d\theta/ds$, *we have*

$$k(t) = \frac{d}{ds}\left(\tan^{-1}\frac{y'(t)}{x'(t)}\right) = \frac{1}{|\alpha'(t)|}\frac{d}{dt}\left(\tan^{-1}\frac{y'(t)}{x'(t)}\right) \quad (\text{assume } x'(t) \neq 0)$$
$$= \frac{1}{|\alpha'(t)|}\frac{1}{1 + \left(\frac{y'(t)}{x'(t)}\right)^2}\frac{x'(t)\, y''(t) - y'(t)\, x''(t)}{(x'(t))^2} = \frac{x'(t)\, y''(t) - y'(t)\, x''(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}.$$

In case $x'(t) \neq 0$, then we have $y'(t) \neq 0$ and can use the identity

$$k(t) = \frac{d}{ds}\left(\cot^{-1}\frac{x'(t)}{y'(t)}\right) = \frac{x'(t)\,y''(t) - y'(t)\,x''(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}.$$

Remark 1.95 ***(Be careful.)*** *By (82), a regular plane curve $\alpha(t) = (x(t), y(t))$ parametrized by an arbitrary parameter t satisfies*

$$k(t_0) = 0 \text{ if and only if } x'(t_0)\,y''(t_0) - y'(t_0)\,x''(t_0) = 0 \text{ (i.e., } \alpha''(t_0) \text{ has no normal component).}$$

*Be careful that $k(t_0) = 0$ is **not** equivalent to $\alpha''(t_0) = 0$. On the other hand, if a regular plane curve $\alpha(s) = (x(s), y(s))$ is parametrized by arc length parameter s, then*

$$k(s_0) = 0 \text{ if and only if } \alpha''(s_0) = (x''(s_0), y''(s_0)) = (0,0),$$

where we note that $\alpha''(s)$ is always pointing in the normal direction (due to $\langle \alpha''(s), \alpha'(s)\rangle = 0$ for all s).

Example 1.96 *Consider the curve $\alpha(t) = (t^3, t^3)$, $t \in (1,\infty)$. Its trace is a half-line in $\mathbb{R}^2$ and has $k(t) \equiv 0$ for all $t \in (1,\infty)$, but $\alpha''(t) = (6t, 6t) \neq 0$ for all $t \in (1,\infty)$.*

Example 1.97 *Consider the graphic curve $\alpha(x) = (x, f(x))$, $x \in I$, then $k(x_0) = 0$ if and only if $f''(x_0) = 0$ if and only if $\alpha''(x_0) = (0,0)$.*

1.5.10 Signed Curvature for Plane Curves Parametrized by Polar Coordinates.

Let (r,θ) be the polar coordinates in $\mathbb{R}^2$, where $r \in (0,\infty)$ and $\theta \in (a,b)$ (some open interval). Let $\alpha(\theta) : (a,b) \to \mathbb{R}^2$ be a regular plane curve parametrized by

$$\alpha(\theta) = (x(\theta), y(\theta)) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad \theta \in (a,b),$$

where now $r(\theta) > 0$ is some positive differentiable function depending on θ. Its arc length L over $\theta \in (a,b)$ is given by

$$L = \int_a^b |\alpha'(\theta)|\,d\theta,$$

where by

$$\begin{aligned}\alpha'(\theta) &= (x'(\theta), y'(\theta)) = r'(\theta)(\cos\theta, \sin\theta) + r(\theta)(-\sin\theta, \cos\theta)\\ &:= \underbrace{r'(\theta)V(\theta) + r(\theta)W(\theta)}, \quad V(\theta)\cdot W(\theta) = 0, \quad V'(\theta) = W(\theta), \quad W'(\theta) = -V(\theta),\end{aligned}$$

we have

$$L = \int_a^b |\alpha'(\theta)|\,d\theta = \int_a^b \sqrt{(r(\theta))^2 + (r'(\theta))^2}\,d\theta. \tag{83}$$

For its signed curvature, we compute

$$\begin{aligned}\alpha''(\theta) &= (x''(\theta), y''(\theta)) = r''(\theta)V(\theta) + r'(\theta)[V'(\theta) + W(\theta)] + r(\theta)W'(\theta)\\ &= \underbrace{(r''(\theta) - r(\theta))V(\theta) + 2r'(\theta)W(\theta)}\end{aligned}$$

and note that

$$\begin{aligned}&x'(\theta)\,y''(\theta) - x''(\theta)\,y'(\theta)\\ &= \det\begin{pmatrix} x'(\theta) & x''(\theta)\\ y'(\theta) & y''(\theta)\end{pmatrix} = \det(\alpha'(\theta), \alpha''(\theta)) \text{ (both } \alpha'(\theta) \text{ and } \alpha''(\theta) \text{ are column vectors)}\end{aligned}$$

In the following we view $V(\theta) = (\cos\theta, \sin\theta)$ and $W(\theta) = (-\sin\theta, \cos\theta)$ as column vector and by

$$\det(V(\theta), W(\theta)) = 1, \quad \det(V(\theta), W'(\theta)) = \det(V'(\theta), W(\theta)) = 0, \quad \det(W(\theta), W'(\theta)) = -1$$

we have

$$\begin{aligned}\det(\alpha'(\theta), \alpha''(\theta)) &= \det\Big(\underbrace{r'(\theta)V(\theta) + r(\theta)W(\theta)}, \ \underbrace{(r''(\theta) - r(\theta))V(\theta) + 2r'(\theta)W(\theta)}\Big) \\ &= 2(r'(\theta))^2 - r(\theta)(r''(\theta) - r(\theta)).\end{aligned}$$

Therefore, the signed curvature $k(\theta)$ is given by

$$k(\theta) = \frac{x'(\theta)y''(\theta) - y'(\theta)x''(\theta)}{\left((x'(\theta))^2 + (y'(\theta))^2\right)^{3/2}} = \frac{r^2(\theta) + 2(r'(\theta))^2 - r(\theta)r''(\theta)}{\left(r^2(\theta) + (r'(\theta))^2\right)^{3/2}}, \quad \theta \in (a,b). \tag{84}$$

1.5.11 Signed Curvature of Level Curves in $\mathbb{R}^2$.

Let $\alpha(s) = (x(s), y(s)) : I \to \mathbb{R}^2$ be a regular plane curve and assume that it is a **level curve** C of a smooth function $\Psi(x,y) : \mathbb{R}^2 \to \mathbb{R}$ given by $\Psi(x,y) = 0$, i.e. we have

$$\Psi(\alpha(s)) = \Psi(x(s), y(s)) = 0, \quad \forall\, s \in I. \tag{85}$$

One can use the function Ψ to express the curvature of α.

Differentiate with respect to s to get (in the following, all partial derivatives of Ψ are evaluated at $\alpha(s)$)

$$\frac{\partial\Psi}{\partial x}\dot{x}(s) + \frac{\partial\Psi}{\partial y}\dot{y}(s) = \langle\nabla\Psi, \ (\dot{x}(s), \dot{y}(s))\rangle = \langle\nabla\Psi, \ t(s)\rangle = 0, \quad \forall\, s \in I \tag{86}$$

and also

$$\left[\frac{\partial\Psi}{\partial x}\ddot{x}(s) + \frac{\partial\Psi}{\partial y}\ddot{y}(s)\right] + \left(\frac{\partial^2\Psi}{\partial x^2}\dot{x}(s) + \frac{\partial^2\Psi}{\partial x\partial y}\dot{y}(s)\right)\dot{x}(s) + \left(\frac{\partial^2\Psi}{\partial x\partial y}\dot{x}(s) + \frac{\partial^2\Psi}{\partial^2 y}\dot{y}(s)\right)\dot{y}(s) = 0, \quad s \in I$$

which is same as

$$\left[\frac{\partial\Psi}{\partial x}\ddot{x}(s) + \frac{\partial\Psi}{\partial y}\ddot{y}(s)\right] + \underbrace{\frac{\partial^2\Psi}{\partial x^2}\dot{x}^2(s) + 2\frac{\partial^2\Psi}{\partial x\partial y}\dot{x}(s)\dot{y}(s) + \frac{\partial^2\Psi}{\partial^2 y}\dot{y}^2(s)} = 0, \quad s \in I \tag{87}$$

or in inner product and matrix multiplication form:

$$\langle(\ddot{x}(s), \ddot{y}(s)), \ \nabla\Psi\rangle + \underbrace{\left\langle\begin{pmatrix}\dot{x}(s)\\ \dot{y}(s)\end{pmatrix}, \ H(s)\begin{pmatrix}\dot{x}(s)\\ \dot{y}(s)\end{pmatrix}\right\rangle} = 0, \quad s \in I \tag{88}$$

where

$$H(s) = \begin{pmatrix}\frac{\partial^2\Psi}{\partial x^2}(x(s), y(s)) & \frac{\partial^2\Psi}{\partial x\partial y}(x(s), y(s)) \\ \frac{\partial^2\Psi}{\partial x\partial y}(x(s), y(s)) & \frac{\partial^2\Psi}{\partial^2 y}(x(s), y(s))\end{pmatrix}, \quad s \in I \tag{89}$$

is the 2×2 **Hessian matrix** of Ψ evaluated at $(x(s), y(s))$.

Let $t(s) = (\dot{x}(s), \dot{y}(s))$ be the unit tangent vector to α at s (as usual, we assume that as s is increasing, the curve goes along the counterclockwise direction). The unit normal vector to C at s is given by $n(s) = (-\dot{y}(s), \dot{x}(s))$. By (86) we know that $\nabla\Psi(x(s), y(s))$ is **perpendicular** to $t(s)$ everywhere and without loss of generality (replace the equation $\Psi(x,y) = 0$ by $-\Psi(x,y) = 0$ if necessary), we may assume that

$$n(s) = (-\dot{y}(s), \dot{x}(s)) = -\frac{\nabla\Psi(\alpha(s))}{|\nabla\Psi(\alpha(s))|} = \frac{-1}{|\nabla\Psi((\alpha(s)))|}\left(\frac{\partial\Psi}{\partial x}(\alpha(s)), \ \frac{\partial\Psi}{\partial y}(\alpha(s))\right) \tag{90}$$

so that $\{t(s), \ -\nabla\Psi/|\nabla\Psi|\}$ has **positive** orientation.

Remark 1.98 *The above assumption means that if C is oriented in the counterclockwise direction and is a simple closed curve, then $\nabla\Psi$ is pointing* ***outside*** *α.*

By definition, the **signed curvature** of α is given by

$$\begin{aligned} k(s) &= \langle t'(s), n(s)\rangle \\ &= \left\langle (\ddot{x}(s), \ddot{y}(s)),\ -\frac{\nabla\Psi(\alpha(s))}{|\nabla\Psi(\alpha(s))|}\right\rangle = \frac{1}{|\nabla\Psi(\alpha(s))|}\langle (\ddot{x}(s), \ddot{y}(s)),\ -\nabla\Psi(\alpha(s))\rangle \end{aligned}$$

and by (87) we obtain

$$k(s) = \frac{1}{|\nabla\Psi(\alpha(s))|}\left(\underbrace{\frac{\partial^2\Psi}{\partial x^2}\dot{x}^2(s) + 2\frac{\partial^2\Psi}{\partial x\partial y}\dot{x}(s)\dot{y}(s) + \frac{\partial^2\Psi}{\partial^2 y}\dot{y}^2(s)}\right). \tag{91}$$

Now by (90) we can write (91) as

$$k(s) = \frac{1}{|\nabla\Psi(\alpha(s))|^3}\left\{\frac{\partial^2\Psi}{\partial x^2}\left(\frac{\partial\Psi}{\partial y}\right)^2 - 2\frac{\partial^2\Psi}{\partial x\partial y}\frac{\partial\Psi}{\partial x}\frac{\partial\Psi}{\partial y} + \frac{\partial^2\Psi}{\partial^2 y}\left(\frac{\partial\Psi}{\partial x}\right)^2\right\} \tag{92}$$

where $\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial x}(\alpha(s))$, $\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial^2\Psi}{\partial x^2}(\alpha(s))$, etc.

We conclude the following:

Lemma 1.99 *Assume C is a regular plane curve given by the equation*

$$\Psi(x, y) = 0 \tag{93}$$

then at $p = (x_0, y_0) \in C$, its ***signed curvature*** *$k(p)$ is given by*

$$k(p) = \frac{1}{|\nabla\Psi(p)|^3}\left\{\left(\frac{\partial\Psi}{\partial y}(p)\right)^2\frac{\partial^2\Psi}{\partial x^2}(p) - 2\frac{\partial\Psi}{\partial x}(p)\frac{\partial\Psi}{\partial y}(p)\frac{\partial^2\Psi}{\partial x\partial y}(p) + \left(\frac{\partial\Psi}{\partial x}(p)\right)^2\frac{\partial^2\Psi}{\partial^2 y}(p)\right\}. \tag{94}$$

Remark 1.100 *(**Important.**) Note that formula (94) is* ***independent of*** *the parametrization of $C = (x(s), y(s))$, $s \in I$. Also* ***the signed curvature is given by the elegant formula:***

$$\begin{aligned} &k(p) \\ &= \operatorname{div}\left(\frac{\nabla\Psi}{|\nabla\Psi|}\right)(p) \quad \textit{(the \textbf{divergence} of the normalized gradient vector field } \frac{\nabla\Psi}{|\nabla\Psi|}\textit{)} \\ &= \frac{\partial}{\partial x}\left(\frac{1}{|\nabla\Psi|}\frac{\partial\Psi}{\partial x}\right)(p) + \frac{\partial}{\partial y}\left(\frac{1}{|\nabla\Psi|}\frac{\partial\Psi}{\partial y}\right)(p). \end{aligned} \tag{95}$$

You can check it by yourself.

Example 1.101 *As an trivial example, we take $\Psi(x, y) = x^2 + y^2 - 1$. Then C is the unit circle centered at $(0, 0)$ and if we parametrize it in the* ***counterclockwise direction****, then the identity (90) holds and the signed curvature k is given by (94). We have*

$$k(p) = \frac{1}{\left(\sqrt{(2x)^2 + (2y)^2}\right)^3}\left\{2(2y)^2 + 2(2x)^2\right\} = 1, \quad \forall\ p \in C. \tag{96}$$

Example 1.102 *In the special case when* $\Psi(x,y) = f(x) - y$, *then if we assume the curve is moving in the direction of increasing* x, *the identity (90) holds. At the point* $p = (x_0, y_0) = (x_0, f(x_0)) \in C$, *(94) can be used and we obtain*

$$k(p) = \frac{1}{|\nabla\Psi(p)|^3}\left\{\frac{\partial^2\Psi}{\partial x^2}(p)\left(\frac{\partial\Psi}{\partial y}(p)\right)^2\right\} = \frac{f''(x_0)}{\left(1 + (f'(x_0))^2\right)^{3/2}} \tag{97}$$

which coincides with the traditional formula.

Example 1.103 *Consider the ellipse* C *given by the equation*

$$\Psi(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

where a, b *are positive constants. One can parametrize the curve* C *as*

$$\alpha(t) = (x(t), y(t)) = (a\cos t, b\sin t), \quad t \in [0, 2\pi].$$

As t *is increasing,* C *is going in the counterclockwise direction and the signed curvature* $k(t)$ *of* $\alpha(t)$ *can be computed by*

$$k(t) = \frac{x'(t)\,y''(t) - y'(t)\,x''(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}} = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}} > 0, \quad t \in [0, 2\pi]. \tag{98}$$

On the other hand, since the identity (90) holds, one can also use the formula (94) to get

$$\begin{aligned}
&k(p) \\
&= \frac{1}{|\nabla\Psi(p)|^3}\left\{\left(\frac{\partial\Psi}{\partial y}(p)\right)^2\frac{\partial^2\Psi}{\partial x^2}(p) - 2\frac{\partial\Psi}{\partial x}(p)\frac{\partial\Psi}{\partial y}(p)\frac{\partial^2\Psi}{\partial x\partial y}(p) + \left(\frac{\partial\Psi}{\partial x}(p)\right)^2\frac{\partial^2\Psi}{\partial^2 y}(p)\right\} \\
&= \frac{1}{\left(\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2\right)^{3/2}}\left\{\left(\frac{2y}{b^2}\right)^2\frac{2}{a^2} + \left(\frac{2x}{a^2}\right)^2\frac{2}{b^2}\right\} = \frac{1}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{3/2}}\frac{1}{a^2b^2}\left(\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right) \\
&= \frac{1}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{3/2}}\frac{ab}{(a^2b^2)^{3/2}} = \frac{ab}{\left(\frac{b^2}{a^2}x^2 + \frac{a^2}{b^2}y^2\right)^{3/2}}, \quad p = (x,y) \in C
\end{aligned} \tag{99}$$

We see that both (98) and (99) are the same.

Exercise 1.104 *Do exercise 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,12, 13, 14 in p. 23 (some problems are quite straightforward).*

Example 1.105 ***(Mapping a plane curve to a space curve by*** $z = f(x,y)$*.)* *(Put this as a homework problem ...) Let* $\alpha(t) = (x(t), y(t))$, $t \in (a,b)$, *be a plane curve and* $z = f(x,y)$ *is a smooth graph with domain* $U \subset \mathbb{R}^2$ *containing the image of* $\alpha(t)$. *Hence* $\alpha(t)$ *is mapped into a space curve* $\beta(t) = (x(t), y(t), z(t))$, *where* $z(t) = f(x(t), y(t))$. *We want to find the relation between the* ***signed curvature*** $k_\alpha(t) \in (-\infty, \infty)$ *of* $\alpha(t)$ *and the* ***curvature*** $k_\beta(t) \in [0, \infty)$ *of* $\beta(t)$. *We have*

$$k_\alpha(t) = \frac{x'(t)\,y''(t) - y'(t)\,x''(t)}{\left((x'(t))^2 + (y'(t))^2\right)^{3/2}}$$

and

$$k_\beta(t) = \frac{|\beta'(t) \wedge \beta''(t)|}{|\beta'(t)|^3}, \quad \beta(t) = (x(t), y(t), f(x(t), y(t))),$$

where

$$\beta'(t) = x'(t)\begin{pmatrix}1\\0\\f_x\end{pmatrix} + y'(t)\begin{pmatrix}0\\1\\f_y\end{pmatrix} = \begin{pmatrix}x'(t)\\y'(t)\\x'(t)\,f_x + y'(t)\,f_y\end{pmatrix} \tag{100}$$

with $f_x = f_x(x(t), y(t))$, $f_y = f_y(x(t), y(t))$, *and then*

$$|\beta'(t)|^3 = \langle \beta'(t), \beta'(t)\rangle^{3/2} = \left[(x'(t))^2\left(1+f_x^2\right) + 2x'(t)\,y'(t)\,f_x f_y + (y'(t))^2\left(1+f_y^2\right)\right]^{3/2} \tag{101}$$

We also have

$$\beta''(t)$$

$$= x''(t)\begin{pmatrix}1\\0\\f_x\end{pmatrix} + y''(t)\begin{pmatrix}0\\1\\f_y\end{pmatrix} + x'(t)\begin{pmatrix}0\\0\\x'(t)\,f_{xx} + y'(t)\,f_{xy}\end{pmatrix} + y'(t)\begin{pmatrix}0\\0\\x'(t)\,f_{yx} + y'(t)\,f_{yy}\end{pmatrix}$$

Hence

$$\beta'(t) \wedge \beta''(t) = I + II, \tag{102}$$

where

$$I = x'(t)\,y''(t)\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix} + (x'(t))^2\begin{pmatrix}0\\-x'(t)\,f_{xx} - y'(t)\,f_{xy}\\0\end{pmatrix} + x'(t)\,y'(t)\begin{pmatrix}0\\-x'(t)\,f_{yx} - y'(t)\,f_{yy}\\0\end{pmatrix}$$

and

$$II = y'(t)\,x''(t)\begin{pmatrix}f_x\\f_y\\-1\end{pmatrix} + y'(t)\,x'(t)\begin{pmatrix}x'(t)\,f_{xx} + y'(t)\,f_{xy}\\0\\0\end{pmatrix} + (y'(t))^2\begin{pmatrix}x'(t)\,f_{yx} + y'(t)\,f_{yy}\\0\\0\end{pmatrix},$$

where $f_x = f_x(x(t), y(t))$, *etc. Thus*

$$\beta'(t) \wedge \beta''(t) = [x'(t)\,y''(t) - y'(t)\,x''(t)]\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix} + M(t)\begin{pmatrix}y'(t)\\-x'(t)\\0\end{pmatrix}, \tag{103}$$

where $(-f_x, -f_y, 1)$ *is* ***normal*** *(pointing upward) to the graph* $z = f(x, y)$ *at* $(x(t), y(t), f(x(t), y(t)))$ *and* $(y'(t), -x'(t), 0)$ *is normal to the plane curve* α *at* $\alpha(t)$, *and* $M(t)$ *comes from the* ***Hessian*** $H_f(x, y)$ *of the function* $f(x, y)$, *given by*

$$M(t) = (x'(t))^2 f_{xx} + 2x'(t)\,y'(t)\,f_{xy} + (y'(t))^2 f_{yy} = \left\langle \begin{pmatrix}x'(t)\\y'(t)\end{pmatrix}, \begin{pmatrix}f_{xx} & f_{xy}\\f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix}x'(t)\\y'(t)\end{pmatrix}\right\rangle.$$

We conclude

$$|\beta'(t) \wedge \beta''(t)|^2$$

$$= \left\{\begin{array}{l}[x'(t)\,y''(t) - y'(t)\,x''(t)]^2\left[1+f_x^2+f_y^2\right]\\ +2M(t)\,[x'(t)\,y''(t) - y'(t)\,x''(t)]\,[x'(t)\,f_y - y'(t)\,f_x] + M^2(t)\left[(x'(t))^2 + (y'(t))^2\right],\end{array}\right. \tag{104}$$

and get the final identity:

$$k_\beta^2(t) = \frac{|\beta'(t) \wedge \beta''(t)|^2}{|\beta'(t)|^6}$$

$$= \frac{\left\{\begin{array}{l}[x'(t)\,y''(t) - y'(t)\,x''(t)]^2\left[1+f_x^2+f_y^2\right]\\ +2M(t)\,[x'(t)\,y''(t) - y'(t)\,x''(t)]\,[x'(t)\,f_y - y'(t)\,f_x] + M^2(t)\left[(x'(t))^2 + (y'(t))^2\right]\end{array}\right\}}{\left[(x'(t))^2\left(1+f_x^2\right) + 2x'(t)\,y'(t)\,f_x f_y + (y'(t))^2\left(1+f_y^2\right)\right]^3}. \tag{105}$$

In case $\alpha(t)$ is parametrized by arc length, i.e., $t = s$, we have

$$|\beta'(s) \wedge \beta''(s)|^2 = \underbrace{k_\alpha^2(s)\left[1 + f_x^2 + f_y^2\right] + 2k_\alpha(s) M(s)\left[x'(s) f_y - y'(s) f_x\right] + M^2(s)},$$

where $f_x = f_x(x(s), y(s))$, $f_{xx} = f_{xx}(x(s), y(s))$, ..., etc, and then

$$k_\beta^2(s) = \underbrace{\frac{k_\alpha^2(s)\left[1 + f_x^2 + f_y^2\right] + 2k_\alpha(s) M(s)\left[x'(s) f_y - y'(s) f_x\right] + M^2(s)}{\left(1 + \left[x'(s) f_x + y'(s) f_y\right]^2\right)^3}} \tag{106}$$

with

$$M(s) = (x'(s))^2 f_{xx} + 2x'(s) y'(s) f_{xy} + (y'(s))^2 f_{yy}. \tag{107}$$

At the point $(x(s), y(s))$ with

$$f_x(x(s), y(s)) = f_y(x(s), y(s)) = 0, \tag{108}$$

the above is reduced to

$$k_\beta^2(s) = \underbrace{k_\alpha^2(s) + M^2}. \tag{109}$$

Another case is when $\alpha(s)$ is a line in $\mathbb{R}^2$ with $(x'(s), y'(s)) = (a, b)$, a unit vector, then $k_\alpha(s) \equiv 0$ and we get

$$k_\beta^2(s) = \frac{\left(a^2 f_{xx} + 2ab f_{xy} + b^2 f_{yy}\right)^2}{\left(1 + (a f_x + b f_y)^2\right)^3}. \tag{110}$$

Example 1.106 ***(Mapping a plane curve to another plane curve by $F(x, y)$.)*** (Put this as a homework problem ...) *Let $\alpha(s) = (x(s), y(s))$, $s \in (a, b)$, be a plane curve parametrized by arc length parameter s and let*

$$F(x, y) = (f(x, y), g(x, y)) : \mathbb{R}^2 \to \mathbb{R}^2$$

be a smooth map and $\beta(s) = F(x(s), y(s))$. We want to find the relation between the ***signed curvature*** *of $\alpha(s)$ and the* ***signed curvature*** *of $\beta(s)$ We have*

$$k_\alpha(s) = x'(s) y''(s) - y'(s) x''(s)$$

and by $\beta(s) = (f(x(s), y(s)), g(x(s), y(s)))$ (denote it as $(p(s), q(s))$), we have

$$k_\beta(s) = \frac{p'(s) q''(s) - q'(s) p''(s)}{\left((p'(s))^2 + (q'(s))^2\right)^{3/2}} = \frac{1}{\left((p'(s))^2 + (q'(s))^2\right)^{3/2}} \det\begin{pmatrix} p'(s) & p''(s) \\ q'(s) & q''(s) \end{pmatrix}. \tag{111}$$

Note that

$$\begin{pmatrix} p'(s) \\ q'(s) \end{pmatrix} = \begin{pmatrix} x'(s) f_x + y'(s) f_y \\ x'(s) g_x + y'(s) g_y \end{pmatrix} = \begin{pmatrix} \langle t(s), \nabla f \rangle \\ \langle t(s), \nabla g \rangle \end{pmatrix},$$

with

$$\begin{aligned} &(p'(s))^2 + (q'(s))^2 \\ &= \langle t(s), \nabla f \rangle^2 + \langle t(s), \nabla g \rangle^2 = \left[x'(s) f_x + y'(s) f_y\right]^2 + \left[x'(s) g_x + y'(s) g_y\right]^2 \\ &= (x'(s))^2 \left[f_x^2 + g_x^2\right] + 2x'(s) y'(s)\left[f_x f_y + g_x g_y\right] + (y'(s))^2 \left[f_y^2 + g_y^2\right], \end{aligned}$$

where $t(s) = (x'(s), y'(s))$ and ∇f, ∇g are both evaluated at $(x(s), y(s))$. Also by

$$\begin{pmatrix} p''(s) \\ q''(s) \end{pmatrix} = \begin{pmatrix} \langle k_\alpha(s) n(s), \nabla f \rangle + \left\langle t(s), \frac{d}{ds} \nabla f \right\rangle \\ \langle k_\alpha(s) n(s), \nabla g \rangle + \left\langle t(s), \frac{d}{ds} \nabla g \right\rangle \end{pmatrix}, \quad t'(s) = k_\alpha(s) n(s), \quad n(s) = (-y'(s), x'(s))$$

and (denote the 2×2 Hessian matrix of f at $(x(s),y(s))$ as $H_f(x(s),y(s))$)

$$\begin{aligned}&\left\langle t(s),\ \frac{d}{ds}\nabla f\right\rangle\\ &=\left\langle\left(\begin{array}{c}x'(s)\\ y'(s)\end{array}\right),\ H_f(x(s),y(s))\left(\begin{array}{c}x'(s)\\ y'(s)\end{array}\right)\right\rangle:=M_f(x(s),y(s))\end{aligned}$$

and

$$\left\langle t(s),\ \frac{d}{ds}\nabla g\right\rangle=\left\langle\left(\begin{array}{c}x'(s)\\ y'(s)\end{array}\right),\ H_g(x(s),y(s))\left(\begin{array}{c}x'(s)\\ y'(s)\end{array}\right)\right\rangle:=M_g(x(s),y(s)),$$

we obtain

$$\left(\begin{array}{c}p''(s)\\ q''(s)\end{array}\right)=\left(\begin{array}{c}\langle k_\alpha(s)n(s),\nabla f\rangle+M_f(x(s),y(s))\\ \langle k_\alpha(s)n(s),\nabla g\rangle+M_g(x(s),y(s))\end{array}\right)$$

and then

$$\begin{aligned}&\det\left(\begin{array}{cc}p'(s)&p''(s)\\ q'(s)&q''(s)\end{array}\right)=\det\left(\begin{array}{cc}\langle t(s),\nabla f\rangle&\langle k_\alpha(s)n(s),\nabla f\rangle+M_f(x(s),y(s))\\ \langle t(s),\nabla g\rangle&\langle k_\alpha(s)n(s),\nabla g\rangle+M_g(x(s),y(s))\end{array}\right)\\ &=\langle t(s),\nabla f\rangle\left[\langle k_\alpha(s)n(s),\nabla g\rangle+M_g\right]-\langle t(s),\nabla g\rangle\left[\langle k_\alpha(s)n(s),\nabla f\rangle+M_f\right]\\ &=k_\alpha(s)\underbrace{\left[\langle t(s),\nabla f\rangle\langle n(s),\nabla g\rangle-\langle t(s),\nabla g\rangle\langle n(s),\nabla f\rangle\right]}+\langle t(s),\ M_g\nabla f-M_f\nabla g\rangle.\end{aligned}\tag{112}$$

We can simplify the first term in (112) as

$$\begin{aligned}&\underbrace{\langle t(s),\nabla f\rangle\langle n(s),\nabla g\rangle-\langle t(s),\nabla g\rangle\langle n(s),\nabla f\rangle}\\ &=\left[x'(s)f_x+y'(s)f_y\right]\left[-y'(s)g_x+x'(s)g_y\right]-\left[x'(s)g_x+y'(s)g_y\right]\left[-y'(s)f_x+x'(s)f_y\right]\\ &=\left\{\begin{array}{l}x'(s)f_x\left[-y'(s)g_x+x'(s)g_y\right]+y'(s)f_y\left[-y'(s)g_x+x'(s)g_y\right]\\ -x'(s)g_x\left[-y'(s)f_x+x'(s)f_y\right]-y'(s)g_y\left[-y'(s)f_x+x'(s)f_y\right]\end{array}\right.\\ &=\left((x'(s))^2+(y'(s))^2\right)(f_xg_y-f_yg_x)=f_xg_y-f_yg_x\end{aligned}$$

and conclude

$$\begin{aligned}&\det\left(\begin{array}{cc}p'(s)&p''(s)\\ q'(s)&q''(s)\end{array}\right)\\ &=k_\alpha(s)(f_xg_y-f_yg_x)+\langle t(s),\ M_g\nabla f-M_f\nabla g\rangle.\end{aligned}\tag{113}$$

The final result is

$$k_\beta(s)=\frac{k_\alpha(s)(f_xg_y-f_yg_x)+\langle t(s),\ M_g\nabla f-M_f\nabla g\rangle}{\left[\langle t(s),\nabla f\rangle^2+\langle t(s),\nabla g\rangle^2\right]^{3/2}},\quad s\in(a,b).\tag{114}$$

<u>End of the Curve Part on 2023/19/26</u>

<u>Move to Surface Part on 2023/10/3</u>

2 Chapter 2: Regular Surfaces.

Remark 2.1 *You may have to read the Appendix of this chapter in p.120 first, if you have forgotten some stuff in Advanced Calculus.*

2.1 Introduction.

Read textbook p. 53 by yourself.

2.2 Regular Surfaces; Inverse Images of Regular Values (this is Section 2.2 of the textbook).

Unlike the case of regular curves, regular surfaces are defined as **sets** in $\mathbb{R}^3$ rather than **maps** (we pay more attention to the shapes of surfaces in $\mathbb{R}^3$ than curves !!!). However, to study their geometric properties, we still need to **parametrize them** (at least locally) and use parametrization $\mathbf{x}(u,v)$ to carry out computations for **mean curvature** and **Gauss curvature**. More precisely, we use parametrization $\mathbf{x}(u,v)$ to perform **CALCULUS** on the surface.

In this chapter, any subset $S \subset \mathbb{R}^3$ has the structure of a **topological space**. Its topology is **induced** from $\mathbb{R}^3$ and we call it **subspace topology**. A subset $A \subset S$ is called an **open set** in S if there is an open set V in $\mathbb{R}^3$ such that $A = V \bigcap S$. However, one should be more careful for **metric structure** on S (in particular when S is a **regular surface**). For any two points $p,\ q \in S$, one can talk about their **Euclidean distance** $|p-q|$ in $\mathbb{R}^3$. On the other hand, one can also talk about their **distance along the surface** S. We shall discuss this in the future.

Roughly speaking, a regular surface in $\mathbb{R}^3$ is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that there are no sharp points, edges, or self-intersections and so that it makes sense to speak of a **tangent plane** at every point of it.

Our goal is to do calculus on a two-dimensional regular surface and this is exactly similar to what we have done on curves.

Definition 2.2 *A subset $S \subset \mathbb{R}^3$ is called a* ***regular surface*** *if for each $p \in S$ there exists a neighborhood (i.e., open set) V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \bigcap S$, where U is an open set in $\mathbb{R}^2$, such that*

1. $\mathbf{x}$ is **differentiable**, i.e., if we write

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$

   then the functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ have continuous partial derivatives of all orders in U.

2. $\mathbf{x} : U \to V \bigcap S$ is a **homeomorphism**. That is, $\mathbf{x} : U \to V \bigcap S$ is continuous (by 1., this is automatically true) and the inverse $\mathbf{x}^{-1} : V \bigcap S \to U$ is also continuous. Note: any subset $M \subset \mathbb{R}^3$ is itself a **topological space** with **subspace topology** induced from the topology of $\mathbb{R}^3$. Therefore, $\mathbf{x} : U \to V \bigcap S$ is a **homeomorphism** between two topological spaces. Also see Remark 2.5 and 2.6 below.

3. For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is **one-one** (some textbook uses the terminology "**nonsingular**") (same as $\ker(d\mathbf{x}_q) = \{0\}$ or $rank\,(d\mathbf{x}_q) = 2$).

Remark 2.3 *Give the "figure 8" example in $\mathbb{R}^2$ to explain that it does not satisfy condition 2 in the above definition. Use both topology and sequence methods to explain it ...*

Remark 2.4 *The map $\mathbf{x} : U \to V \bigcap S$ is called a* ***(local) parametrization*** *or* ***a system of local coordinates*** *in (a neighborhood of) p. The set $V \bigcap S \subset S$ is called a* ***coordinate neighborhood*** *of p on S.*

Remark 2.5 *If the map* $\mathbf{x}^{-1}: V\bigcap S\to U$ *(it is one-one and onto) is the* ***restriction*** *of a continuous map* $F: V\subset\mathbb{R}^3\to\mathbb{R}^2$*, then* $\mathbf{x}^{-1}: V\bigcap S\to U$ *is continuous. This is because for any open set* $N\subset U$ *we have*

$$\left(\mathbf{x}^{-1}\right)^{-1}(N)=\mathbf{x}(N)=F^{-1}(N)\bigcap S$$

and $F^{-1}(N)$ *is open in* V *and it is also open in* $\mathbb{R}^3$ *since* V *is open in* $\mathbb{R}^3$.

Remark 2.6 *The map* $\mathbf{x}^{-1}: V\bigcap S\to U$ *is continuous is* ***equivalent to*** *the fact that for each* $p\in V\bigcap S$ *and each sequence* $p_n\in V\bigcap S$ *with* $p_n\to p$ *in* $V\bigcap S$ *(this means the Euclidean distance* $|p_n-p|$ *in* $\mathbb{R}^3$ *converges to* 0 *as* $n\to\infty$*), we have*

$$\lim_{n\to\infty}\mathbf{x}^{-1}(p_n)=\mathbf{x}^{-1}(p). \tag{115}$$

In Topology, there is a theorem which says the following: Let $f: X\to Y$ *be a map (both* X *and* Y *are topological spaces). If* f *is a continuous map, then for every convergent sequence* $x_n\to x$ *in* X*, the sequence* $f(x_n)$ *converge to* $f(x)$ *in* Y*. The converse holds if* X *is* ***metrizable.***

Remark 2.7 ***(Important.)*** *Condition 1 is natural if we want to do differential calculus on the surface* S*. Condition 2 is more subtle, which will be explained more later on. At this moment, we can see that a regular surface* ***cannot have self-intersections*** *(why? explain this using curve and sequence argument). Condition 3 guarantees the existence of a* ***tangent plane*** *(we will give a precise definition of it) at every point of* S*. This is similar to the condition that a* ***regular curve*** $\alpha(t)$ *in* $\mathbb{R}^2$ *has* ***nonzero*** *tangent vectors everywhere (hence the existence of tangent lines everywhere), i.e.,* $\alpha'(t)$ *has* ***rank 1*** *for all* t *in the domain of* α*.*

Remark 2.8 *The differential* $d\mathbf{x}_q:\mathbb{R}^2\to\mathbb{R}^3$ *is one-one (equivalent to* $\ker d\mathbf{x}_q=\{0\}$*) implies that its Jacobi matrix (with respect to the standard bases of* $\mathbb{R}^2$ *and* $\mathbb{R}^3$*)* J*, given by*

$$J=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}\end{pmatrix}$$

has ***rank 2*** *(this is due to the* ***Rank Theorem*** *in linear algebra). This means that the two column vectors are* ***linearly independent*** *or there exists a* 2×2 *submatrix* M *of* J *with* $\det M\neq 0$*. These two vectors can span a plane passing through the point* $\mathbf{x}(q)$*.*

Let $q=(u_0,v_0)\in U\subset\mathbb{R}^2$. The vector $e_1=(1,0)$ is tangent to the curve $u\to(u,v_0)$ in $\mathbb{R}^2$ whose image under $\mathbf{x}(u,v)$ is the curve $u\to(x(u,v_0),y(u,v_0),z(u,v_0))$ in $\mathbb{R}^3$. We call it the **coordinate curve** $v=v_0$ on the regular surface S. Its **tangent vector** at $\mathbf{x}(q)$ in $\mathbb{R}^3$ is

$$\frac{\partial\mathbf{x}}{\partial u}(q)=\frac{\partial\mathbf{x}}{\partial u}(u_0,v_0)=\left(\frac{\partial x}{\partial u}(u_0,v_0),\frac{\partial y}{\partial u}(u_0,v_0),\frac{\partial z}{\partial u}(u_0,v_0)\right).$$

Similarly, the **coordinate curve** $u=u_0$ on the regular surface S has a **tangent vector** at $\mathbf{x}(q)$ in $\mathbb{R}^3$ given by

$$\frac{\partial\mathbf{x}}{\partial v}(q)=\frac{\partial\mathbf{x}}{\partial v}(u_0,v_0)=\left(\frac{\partial x}{\partial v}(u_0,v_0),\frac{\partial y}{\partial v}(u_0,v_0),\frac{\partial z}{\partial v}(u_0,v_0)\right).$$

By the definition of differential, we know that

$$d\mathbf{x}_q(e_1)=\frac{\partial\mathbf{x}}{\partial u}(q)=\left(\frac{\partial x}{\partial u}(q),\frac{\partial y}{\partial u}(q),\frac{\partial z}{\partial u}(q)\right),\quad e_1=(1,0).$$

and

$$d\mathbf{x}_q(e_2) = \frac{\partial \mathbf{x}}{\partial v}(q) = \left(\frac{\partial x}{\partial v}(q), \frac{\partial y}{\partial v}(q), \frac{\partial z}{\partial v}(q)\right), \quad e_2 = (0,1).$$

Now the matrix of the linear map $d\mathbf{x}_q$ with respect to the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively is given by

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u}(q) & \frac{\partial x}{\partial v}(q) \\ \frac{\partial y}{\partial u}(q) & \frac{\partial y}{\partial v}(q) \\ \frac{\partial z}{\partial u}(q) & \frac{\partial z}{\partial v}(q) \end{pmatrix}.$$

Since $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ has **rank 2** at $q = (u_0, v_0)$, the image of the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is a **two-dimensional plane** P in $\mathbb{R}^3$ passing through the point $p := \mathbf{x}(u_0, v_0) \in S$ (we will define P as the **tangent plane** of S at p and discuss it more later on). Note that at least one of the following

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad \frac{\partial(x,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} \tag{116}$$

is **not** zero at q.

Example 2.9 *Show that the unit sphere*

$$S^2 = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a regular surface in $\mathbb{R}^3$. *Follow the book's explanation in p. 57, 58, 59.*

We first observe the following:

Proposition 2.10 *Let*

$$S = \left\{(x, y, f(x,y)) : (x,y) \in U \subset \mathbb{R}^2\right\}$$

be the graph of a differentiable function $f(x,y)$ *defined on some open set* U *of* $\mathbb{R}^2$ *(*$U = \mathbb{R}^2$ *is allowed). Then it is a regular surface in* $\mathbb{R}^3$.

Proof. Define the parametrization $\mathbf{x}(u,v) : U \to \mathbb{R}^3$ as

$$\mathbf{x}(u,v) = (u, v, f(u,v)), \quad (u,v) \in U.$$

Then follow the proof as in the book p. 60. To show that the map $\mathbf{x}^{-1} : S \to \mathbb{R}^2$ is continuous, one can use the projection argument in Remark 2.5 or use the sequence argument in Remark 2.6. □

Lemma 2.11 *If* $S \subset \mathbb{R}^3$ *is a regular surface and* $S_1 \subset S$ *is any* ***open subset*** *of* S, *then* $S_1 \subset \mathbb{R}^3$ *is itself a* ***regular surface*** *in* $\mathbb{R}^3$. *In particular, if* $\mathbf{x} : U \to V \bigcap S$ *is a* ***parametrization****, then* $V \bigcap S$ *is itself a regular surface in* $\mathbb{R}^3$.

Remark 2.12 $S_1 \subset S$ *is an* ***open subset*** *of* S *means that there exists an open set* V *in* $\mathbb{R}^3$ *such that* $S_1 = V \bigcap S$.

Proof. Exercise. □

Lemma 2.13 *Assume* $S \subset \mathbb{R}^3$ *satisfies* $S = \bigcup_{i \in J} S_i$ *for some index set* J, *where each* S_i *is a regular surface in* $\mathbb{R}^3$ *and each* S_i *is an* ***open subset*** *of* S. *Then* $S \subset \mathbb{R}^3$ *is a regular surface.*

Remark 2.14 *The condition that each* S_i *is an* ***open subset*** *of* S *is essential. Give a counter example with two regular surfaces* $S_1,\ S_2$ *touching at a point* $p \in S_1 \bigcap S_2$.

Proof. Exercise. □

Lemma 2.15 *Let O_1 and O_2 be two open set in $\mathbb{R}^3$ and $\varphi : O_1 \to O_2$ is a* ***diffeomorphism****. If $S_1 \subset O_1$ is a regular surface, then $S_2 = \varphi(S_1) \subset O_2$ is also a* ***regular surface*** *in $\mathbb{R}^3$.*

Proof. The key point is to observe that if $\mathbf{x} : U \to \mathbf{x}(U) \subset S_1$ is a parametrization of S_1, then $\varphi \circ \mathbf{x} : U \to S_2$ is a parametrization of S_2. The details are left to you. □

Definition 2.16 *Let $U \subset \mathbb{R}^n$ be an open set and $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable map (C^∞ map). A point $p \in U$ is called a* ***critical point*** *of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is* ***not onto*** *(the definition is trivial if $n < m$ since it is impossible for $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ to be* ***onto*** *for any $p \in U$, and so any $p \in U$ is a critical point; therefore* ***we focus on the case*** *$n \geq m$). The* ***image*** *$F(p)$ of a critical point p is called a* ***critical value*** *of F. A point of $\mathbb{R}^m$ which is not a critical value is called a* ***regular value*** *of F. Note: By definition, critical points are in the* ***domain*** *of F and critical values (and regular values) are in the* ***image*** *of F. Note that critical values and regular values of F are* ***points*** *in $\mathbb{R}^m$. They are* ***not*** *numbers (unless $m = 1$).*

Remark 2.17 *By definition, if $q \in \mathbb{R}^m$ is not in the image of $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$, we still call it a* ***(trivial) regular value*** *of F (see textbook p. 60, line -4). In such a case, $F^{-1}(q) = \varnothing$. Therefore, when we discuss the properties of a regular value $q \in \mathbb{R}^m$, we only focus on the case $q \in F(U)$.*

Remark 2.18 ***(Important observation.)*** *In case $m = 1$ in the above definition, then $dF_p : \mathbb{R}^n \to \mathbb{R}$ is* ***not onto*** *is equivalent to*

$$dF_p = 0 \quad on \quad \mathbb{R}^n. \tag{117}$$

Remark 2.19 *By definition, if $q \in \mathbb{R}^m$ is in the image of $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and is a* ***regular value*** *of F, then the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is* ***onto*** *for* ***all*** *p in the* ***nonempty*** *set*

$$F^{-1}(q) = \{p \in U : F(p) = q\}.$$

Example 2.20 *If $f : U \subset \mathbb{R} \to \mathbb{R}$, then $x_0 \in U$ is a* ***critical point*** *if and only if $f'(x_0) = 0$. The differential $df_{x_0}(v) = f'(x_0)v = 0$ is a zero map.*

Example 2.21 *Draw a picture for the case when $f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ and identify its critical values. At a critical point $p \in U$, the tangent plane P to the graph $z = f(x, y)$ at $(p, f(p))$ is given by the horizontal plane $z = f(p)$.*

Example 2.22 *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, then $p \in U$ is a* ***critical point*** *if and only if*

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Hence $a \in f(U)$ is a regular value if and only if f_x, f_y and f_z do not vanish simultaneously ***at any point*** *in the inverse image*

$$f^{-1}(a) = \{p \in U : f(p) = a\}.$$

Our purpose is to observe the following.

Theorem 2.23 *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable map (here U is an open set in $\mathbb{R}^3$) and $a \in f(U)$ is a* ***regular value,*** *then $f^{-1}(a) \subset U$ is a* ***regular surface*** *in $\mathbb{R}^3$.*

Remark 2.24 *However, if $a \in f(U)$ is a* ***critical value,*** *then $f^{-1}(a) \subset U$ may be or may not be a* ***regular surface*** *in $\mathbb{R}^3$.*

Proof. The idea is to use the **inverse function theorem (IVFT)** in advanced calculus (this is book proof; in my opinion, using **implicit function theorem (IMFT)** is more straightforward; see Remark 2.25 below). Let $p = (x_0, y_0, z_0) \in f^{-1}(a)$. Without loss of generality, we may assume that $f_z(p) \neq 0$. Define a map $F : U \subset \mathbb{R}^3$ (with coordinates (x, y, z)) $\to \mathbb{R}^3$ (with coordinates (u, v, t)) $\to \mathbb{R}^3$ by $F(x, y, z) = (x, y, f(x, y, z))$ and note that

$$dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}, \quad f_x = f_x(x_0, y_0, z_0), \ etc.$$

Hence $\det(dF_p) = f_z(p) \neq 0$. By the **inverse function theorem**, there exist neighborhood V of p (where $V \subset U$) and neighborhood W of $F(p)$ such that $F : V \to W$ is a **diffeomorphism** (explain this terminology) and one can solve $(x, y, z) \in V$ as a **differentiable function** (denote it as $F^{-1} : W \to V$) of $(u, v, t) \in W$ by the following

$$x = u, \quad y = v, \quad z = g(u, v, t), \quad (u, v, t) \in W.$$

In particular, $z = g(u, v, a) = g(x, y, a)$ (call it $h(x, y)$) is a differentiable function defined in the projection V_{proj} of V onto the xy-plane. Hence the set $V \bigcap f^{-1}(a) \subset U$ can be expressed as

$$\left\{(x, y, z) \in V \bigcap f^{-1}(a) \subset U\right\} = \left\{(x, y, h(x, y)) \in V \bigcap f^{-1}(a) \subset U\right\},$$

i.e., the graph of $h(x, y)$ over V_{proj} is **precisely** the set $V \bigcap f^{-1}(a)$. More precisely, we have

$$F\left(V \bigcap f^{-1}(a)\right) = W \bigcap \{(u, v, t) : t = a\}.$$

By Proposition 2.10, $V \bigcap f^{-1}(a)$ is a regular surface containing $p \in f^{-1}(a)$. Hence there exist a coordinate neighborhood around p and a local parametrization given by $\mathbf{x}(x, y) = (x, y, h(x, y))$, $(x, y) \in V_{proj}$. Since $p \in f^{-1}(a)$ is arbitrary, the set $f^{-1}(a)$ is a regular surface. $\square$

Remark 2.25 ***(Important.)*** *In the above proof, we solve the equation $f(x, y, z) = a$ to get $z = h(x, y)$ with $z_0 = h(x_0, y_0)$, under the assumption that $f(p) = a$ and $f_z(p) \neq 0$, $p = (x_0, y_0, z_0)$. Since for any $p \in f^{-1}(a)$ (denote $f^{-1}(a)$ as $S \subset \mathbb{R}^3$) there is an open set $V \subset \mathbb{R}^3$ such that $V \bigcap S$ is a* ***graph-like regular surface*** *(of the form $z = h(x, y)$ or $y = h(x, z)$ or $x = h(y, z)$), we can apply Lemma 2.13 to conclude that $S \subset \mathbb{R}^3$ is a regular surface.*

Remark 2.26 *At $p = (x_0, y_0, z_0) \in f^{-1}(a)$, the regular surface (which is a* ***level set****) has its "tangent plane"* ***perpendicular to*** *the gradient vector*

$$\nabla f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0), \ f_y(x_0, y_0, z_0), \ f_z(x_0, y_0, z_0)).$$

This is easy to see. Thus, the ***tangent plane*** *at p of the regular surface $f^{-1}(a)$ is the* ***kernel*** *of the differential $df_p : \mathbb{R}^3 \to \mathbb{R}$. This fact is a special case of a more general theorem in advanced calculus.*

Example 2.27 *Do Example 2 (ellipsoid) in p. 63.*

Definition 2.28 *A regular surface $S \subset \mathbb{R}^3$ is called* ***(path) connected*** *if any $p, \ q \in S$ can be joined by a continuous curve lying on S.*

Remark 2.29 *A regular surface may be or may not be connected.*

Example 2.30 *Do Example 3 in p. 63.*

The following fact will be used often.

Lemma 2.31 *If $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ is a* ***nonzero*** *continuous function and S is connected regular surface, then f does not change sign on S.*

Proof. Assume not. Hence there exist $p,\ q \in S$ such that $f(p) > 0$ and $f(q) < 0$. As S is connected, we can find a continuous curve $\alpha(t) : [a,b] \to S$ with $\alpha(a) = p,\ \alpha(b) = q$. Now the function $f(\alpha(t)) : [a,b] \to \mathbb{R}$ is a continuous function with $f(\alpha(a)) > 0$ and $f(\alpha(b)) < 0$. By the Intermediate Value Theorem, there exists $c \in (a,b)$ such that $f(\alpha(c)) = 0$, where $\alpha(c) \in S$. This gives a contradiction. □

Example 2.32 *Do Example 4 in p. 64. Note that for $\nabla f(x,y,z) = (0,0,0)$, we must have either $(x,y,z) = (0,0,0)$ or*

$$\sqrt{x^2+y^2} = a \quad \text{and} \quad z = 0.$$

Since both cases will not happen, the number r^2 is a regular value and $f^{-1}(r^2)$ is a regular surface.

Example 2.33 *Do Example 6 in p. 67.*

The following important fact says that any regular surface is **locally** the graph of a differentiable function.

Theorem 2.34 *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighborhood V of p in S (i.e., V is an open set in S) such that V is the graph of a differentiable function of the form $z = f(x,y)$ or $y = g(x,z)$ or $x = h(y,z)$.*

Proof. Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbf{x}(U) \subset S$ be a parametrization of S near p with $\mathbf{x}(q) = p$, where $q \in U$ and

$$\mathbf{x}(u,v)) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U.$$

Without loss of generality, we may assume that (note that $rank\,(d\mathbf{x}_q) = 2$)

$$\frac{\partial(x,y)}{\partial(u,v)}(q) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}(q) \neq 0. \tag{118}$$

Consider the map $\pi \circ \mathbf{x} : U \to \mathbb{R}^2$, where π is the projection $\pi(x,y,z) = (x,y)$. Then $\pi(\mathbf{x}(u,v)) = (x(u,v), y(u,v))$ is a differentiable map with **nonzero Jacobian** at q. By the inverse function theorem, there exist **small** neighborhoods V_1 of q in $\mathbb{R}^2$ (without loss of generality, we may assume $V_1 \subset U$) and V_2 of $\pi(\mathbf{x}(q)) = \pi(p)$ in $\mathbb{R}^2$ such that

$$\pi \circ \mathbf{x} : V_1 \subset \mathbb{U} \subset \mathbb{R}^2 \ (uv\text{-space}) \to V_2 \subset \mathbb{R}^2 \ (xy\text{-space})$$

is a **diffeomorphism** with differentiable inverse $(\pi \circ \mathbf{x})^{-1} : V_2 \to V_1$. Since $\mathbf{x} : U \to \mathbf{x}(U) \subset S_1$ is a homeomorphism, the set $V = \mathbf{x}(V_1)$ is a **neighborhood** of p in S and we see that $\pi : V \to V_2$ is one-one and onto.

Now if we compose the map $(\pi \circ \mathbf{x})^{-1} : (x,y) \to (u(x,y), v(x,y))$ with the function $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbf{x}(U) \subset S$, we find that V is the graph of the differentiable function $z = z(u(x,y), v(x,y)) := f(x,y)$ (note that **the three sets V_1, V_2 and V are all in one-one and onto correspondence**) over the open set $(x,y) \in V_2$. More precisely, we have

$$\mathbf{x} \circ (\pi \circ \mathbf{x})^{-1} : V_2 \overset{(\pi\circ\mathbf{x})^{-1}}{\to} V_1 \overset{\mathbf{x}}{\to} V$$

and for $(x,y) \in V_2$, we have

$$\begin{aligned} &\left(\mathbf{x} \circ (\pi \circ \mathbf{x})^{-1}\right)(x,y) \\ &= \mathbf{x}(u(x,y), v(x,y)) = (x(u(x,y), v(x,y)), y(u(x,y), v(x,y)), z(u(x,y), v(x,y))) \\ &= (x, y, z(u(x,y), v(x,y))) \in V, \end{aligned}$$

i.e. V is the graph of a function of the form $z = f(x,y)$ over V_2. The proof of the other two cases are similar. □

Remark 2.35 *From the above proof we observe that* $\mathbf{y} = \mathbf{x} \circ (\pi \circ \mathbf{x})^{-1} : V_2 \to S$ *is a parametrization that covers* $p \in S$ *such that* $\pi \circ \mathbf{y}$ *is the identity map on* V_2*. This says that one can always cover a point* $p \in S$ *by a parametrization that is the inverse map of a projection onto a coordinate plane.*

The next theorem says that if we already know that S is a regular surface and we have a candidate $\mathbf{x}$ for a parametrization, we do not have to check that $\mathbf{x}^{-1}$ is continuous provided the other conditions in the definition are satisfied.

Theorem 2.36 *Let* $p \in S$ *be a point of a* ***regular surface*** S *and let* $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *be a map with* $p \in \mathbf{x}(U) \subset S$ *such that conditions 1 and 3 in Definition 2.2 hold. Assume* $\mathbf{x} : U \to \mathbf{x}(U)$ *is one-one and onto. Then* $\mathbf{x}^{-1} : \mathbf{x}(U) \to U$ *is continuous.*

Proof. This is obvious from the proof of Theorem 2.34. For any $p = \mathbf{x}(q) \in \mathbf{x}(U)$, $q \in U$, we may assume that

$$\frac{\partial (x, y)}{\partial (u, v)}(q) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}(q) \neq 0 \tag{119}$$

and, by the proof of Theorem 2.34, there is a neighborhood $V \subset S$ of p such that it is the graph of a differentiable function $z = f(x, y)$ over an open set V_2 of the xy-plane. Also, there is an open set $V_1 \subset U$ of q in $\mathbb{R}^2$ (uv-space) and V_2 of $\pi(p)$ in $\mathbb{R}^2$ (xy-space) such that $\pi \circ \mathbf{x} : V_1 \to V_2$ is a **diffeomorphism.** Let $V = \mathbf{x}(V_1)$. Note that the map $\pi : V \to V_2$ is continuous, one-one and onto, due to the fact that $\mathbf{x} : V_1 \to V$ is continuous, one-one and onto, and $\pi \circ \mathbf{x} : V_1 \to V_2$ is a diffeomorphism (for any two points $a \neq b \in V$, we have $a = \mathbf{x}(a_1)$ and $b = \mathbf{x}(b_1)$, $a_1 \neq b_1$, and so $\pi(a) = \pi \circ \mathbf{x}(a_1) \neq \pi \circ \mathbf{x}(b_1) = \pi(b)$).

By the above, we have

$$\mathbf{x}^{-1}(\theta) = (\pi \circ \mathbf{x})^{-1} \circ \pi(\theta), \quad \forall\, \theta \in V,$$

which implies that $\mathbf{x}^{-1} : V \to V_1$ is a continuous map since both $(\pi \circ \mathbf{x})^{-1}$ and π are both continuous maps. In particular, the map $\mathbf{x}^{-1}$ is continuous at p, where $p \in \mathbf{x}(U)$ is arbitrary. Therefore, $\mathbf{x}^{-1} : \mathbf{x}(U) \to U$ is a continuous map. The proof is done. □

Remark 2.37 *From the above proof, we can say that, up to a diffeomorphism, the map* $\mathbf{x}^{-1}$ *is like* π*. Hence it is continuous.*

Example 2.38 *(Example 5 in p. 66.) Let* S *be the graph of the function* $z = \sqrt{x^2 + y^2}$, (x, y) *lies in some open set* O *of* $\mathbb{R}^2$ *containing* $(0, 0)$*. We claim that it is* **not** *a regular surface in* $\mathbb{R}^3$*. To see this, by Theorem 2.34 there exists a neighborhood* V *of* $(0, 0, 0)$ *in* S *(i.e.,* V *is an open set in* S*) such that* V *is the graph of a differentiable function of the form* $z = f(x, y)$ *or* $y = g(x, z)$ *or* $x = h(y, z)$*. It is clear that the last two forms are impossible since the projection of* V *onto* xz *and* yz *planes are not one-one. Therefore we must have* $f(x, y) = \sqrt{x^2 + y^2}$*. We get a contradiction since* $\sqrt{x^2 + y^2}$ *is not differentiable at* $(0, 0)$*.*

2.3 Change of Parameters; Differentiable Functions on Surfaces (this is Section 2.3 of the textbook).

Note that each point p on a regular surface S belongs to a coordinate neighborhood. The points of such a neighborhood are characterized by their coordinates. We shall define interesting local properties of S in terms of these coordinates.

We first define the meaning of **a differentiable function** f **on** S (or on an open set of S).

Definition 2.39 *Let* $V \subset S$ *be an open set. A function* $f : V \subset S \to \mathbb{R}$ *(or* $\mathbb{R}^n$*) is called* ***differentiable*** *at* $p \in V$ *if, for* **some** *parametrization* $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ *with* $p \in \mathbf{x}(U) \subset V$*, the function* $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ *(or* $\mathbb{R}^n$*) is* ***differentiable*** *at* $\mathbf{x}^{-1}(p)$*. If* f *is differentiable at every* $p \in V$*, we say it is differentiable on* V*.*

Remark 2.40 *According to the definition, the map* $f=\mathbf{x}^{-1}:\mathbf{x}(U)\subset S\to U$ *is a differentiable map since* $f\circ\mathbf{x}:U\subset\mathbb{R}^2\to U\subset\mathbb{R}^2$ *is an identity map.*

Remark 2.41 *(**Notation convention.**) By abuse of notation, we shall write* $f(\mathbf{x}(u,v))$ *as* $f(u,v)$ *(essentially we are identifying* $(u,v)\in U$ *and* $\mathbf{x}(u,v)\in\mathbf{x}(U)$*) if no confusion occurs.*

To see Definition 2.39 is well-defined, we observe the following:

Theorem 2.42 *(**Change of parameters.**). Let* p *be a point on a regular surface* S *and* $\mathbf{x}:U\subset\mathbb{R}^2\to S$*,* $\mathbf{y}:V\subset\mathbb{R}^2\to S$ *be two parametrizations of* S *such that* $p\in\mathbf{x}(U)\bigcap\mathbf{y}(V)=W$ *(note that* $W\subset S$ *is an open set). Then the change of coordinates map (which is one-one and onto)*

$$h=\mathbf{x}^{-1}\circ\mathbf{y}:\mathbf{y}^{-1}(W)\subset\mathbb{R}^2\to\mathbf{x}^{-1}(W)\subset\mathbb{R}^2 \tag{120}$$

*is a **diffeomorphism** between two open sets in Euclidean space.*

Remark 2.43 *Draw a picture for* h.

Proof. Clearly $h=\mathbf{x}^{-1}\circ\mathbf{y}:\mathbf{y}^{-1}(W)\to\mathbf{x}^{-1}(W)$ is a **homeomorphism** since it is composed of homeomorphisms. Let $r\in\mathbf{y}^{-1}(W)$ and $q=h(r)\in\mathbf{x}^{-1}(W)$. Since $\mathbf{x}=\mathbf{x}(u,v)=(x(u,v),y(u,v),z(u,v))$ is a parametrization, we may assume that

$$\frac{\partial(x,y)}{\partial(u,v)}(q)\neq 0,\quad q\in U.$$

We then extend $\mathbf{x}$ to a map $F:U\times\mathbb{R}\to\mathbb{R}^3$ by

$$F(u,v,t)=(x(u,v),y(u,v),z(u,v)+t),\quad (u,v)\in U,\quad t\in\mathbb{R}.$$

This map is clearly differentiable and $F=\mathbf{x}$ on $U\times\{0\}$, $F(q,0)=\mathbf{x}(q)$, and $F(u,v,t)\in S$ if and only if $t=0$. The function F maps a vertical cylinder C over U into a "vertical cylinder" over $\mathbf{x}(U)$. Moreover, the differential dF_q is **nonsingular** (the matrix for dF_q is now 3×3). By the inverse function theorem, there exists a neighborhood M of $\mathbf{x}(q)$ in $\mathbb{R}^3$ (M is an open set in $\mathbb{R}^3$) such that F^{-1} exists and is differentiable on M (**note that we have extended** $\mathbf{x}^{-1}$ **locally into** F^{-1}, **which is a differentiable map on some open set in** $\mathbb{R}^3$).

By the continuity of $\mathbf{y}$, there is a neighborhood N of r in V such that $\mathbf{y}(N)\subset M$. Note that

$$h|_N=\mathbf{x}^{-1}\circ\mathbf{y}|_N=F^{-1}\circ\mathbf{y}|_N, \tag{121}$$

which is a composition of two differentiable maps (each map has domain an open set in **Euclidean space**). Hence h is differentiable on N and, in particular, differentiable at r. As $r\in\mathbf{y}^{-1}(W)$ is arbitrary, h is differentiable on $\mathbf{y}^{-1}(W)$.

Similarly, we can show that h^{-1} is differentiable on $\mathbf{x}^{-1}(W)$. Hence $h:\mathbf{y}^{-1}(W)\to\mathbf{x}^{-1}(W)$ is a diffeomorphism. □

Remark 2.44 *(**Important.**) From the proof of Theorem 2.42, we see that for any parametrization* $\mathbf{x}:U\to\mathbf{x}(U)\subset S$*, the inverse map* $\mathbf{x}^{-1}:\mathbf{x}(U)\to U$ *on* S *near any point* $p\in\mathbf{x}(U)$ *can be **locally extended to a differentiable map*** F^{-1} ***on some open set*** M ^{3} ***in*** $\mathbb{R}^3$, ***where*** $p\in M$.

Remark 2.45 *By the above theorem, Definition 2.39 **does not depend on** the choice of the parametrization* $\mathbf{x}$*. If* $\mathbf{y}$ *is another parametrization around* $p\in S$*, then* $f\circ\mathbf{y}=f\circ\mathbf{x}\circ h$ *is also differentiable. Draw a picture for this.*

As a consequence of from the proof of Theorem 2.42, we have the following useful result:

Theorem 2.46 *Let $S \subset \mathbb{R}^3$ be a regular surface and let*

$$\Phi : V \text{ (open set)} \subset \mathbb{R}^n \text{ ($n \in \mathbb{N}$ is arbitrary)} \to S$$

be a differentiable map. For fixed $q \in V$, let $p = \Phi(q) \in S$ and let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbf{x}(U) \subset S$ be a parametrization with $p \in \mathbf{x}(U)$. Then the map

$$\mathbf{x}^{-1} \circ \Phi : \Phi^{-1}(\mathbf{x}(U)) \text{ (open set in } \mathbb{R}^n\text{)} \to U \subset \mathbb{R}^2 \tag{122}$$

is ***differentiable*** *on $\Phi^{-1}(\mathbf{x}(U))$.*

Remark 2.47 *The property in Theorem 2.46 can be applied to Example 3 in p. 77. See Example 2.54 below.*

Proof. We may assume $\Phi^{-1}(\mathbf{x}(U)) \neq \varnothing$ and for any $q \in \Phi^{-1}(\mathbf{x}(U))$, we have $p = \Phi(q) \in \mathbf{x}(U) \subset S$. We know that near $p \in \mathbf{x}(U)$, **the inverse map $\mathbf{x}^{-1}$ on S can be locally extended to a differentiable map F^{-1} on some open set in $\mathbb{R}^3$**. By this, we have

$$\left(\mathbf{x}^{-1} \circ \Phi\right)(q) = \left(F^{-1} \circ \Phi\right)(q) \quad \text{for all} \quad q \in \Phi^{-1}(\mathbf{x}(U)). \tag{123}$$

Since $(F^{-1} \circ \Phi)(q)$ is a differentiable map on $\Phi^{-1}(\mathbf{x}(U))$ ($F^{-1} \circ \Phi$ is the composition of two differentiable maps defined on open sets in Euclidean space), the function $\mathbf{x}^{-1} \circ \Phi$ is differentiable on $\Phi^{-1}(\mathbf{x}(U))$. □

Example 2.48 *Do Example 1 in p. 75. Discuss height function and distance function.*

Definition 2.49 *A continuous map $\varphi : V_1 \subset S_1 \to S_2$ from an open set V_1 of a regular surface S_1 into another regular surface S_2 is called* ***differentiable*** *at $p \in V_1$ if, given parametrizations*

$$\mathbf{x}_1 : U_1 \subset \mathbb{R}^2 \to S_1, \qquad \mathbf{x}_2 : U_2 \subset \mathbb{R}^2 \to S_2$$

satisfying $p \in \mathbf{x}_1(U_1)$, $\mathbf{x}_1(U_1) \subset V_1$, and $\varphi(\mathbf{x}_1(U_1)) \subset \mathbf{x}_2(U_2)$, the map

$$\mathbf{x}_2^{-1} \circ \varphi \circ \mathbf{x}_1 : U_1 \to U_2 \tag{124}$$

is differentiable at $q = \mathbf{x}_1^{-1}(p)$.

Remark 2.50 *Again, this definition is independent of the choice of parametrizations.*

Definition 2.51 *A continuous map $\varphi : S_1 \to S_2$ between two regular surfaces is called a* ***diffeomorphism*** *if φ is differentiable with a differentiate inverse $\varphi^{-1} : S_2 \to S_1$. In such a case, these two regular surfaces S_1 and S_2 are said to be* ***diffeomorphic****.*

Remark 2.52 *From the viewpoint of differentiability, two regular surfaces which are diffeomorphic are indistinguishable.*

Example 2.53 *(This is Example 2 in p. 76.) Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization. For any $p \in \mathbf{x}(U)$ and any parametrization $\mathbf{y} : V \subset \mathbb{R}^2 \to S$ around p with $p \in \mathbf{x}(U) \bigcap \mathbf{y}(V) = W$. The change of parameters map $\mathbf{x}^{-1} \circ \mathbf{y} : \mathbf{y}^{-1}(W) \to \mathbf{x}^{-1}(W)$ is differentiable. Therefore, by definition, the inverse map $\mathbf{x}^{-1} : \mathbf{x}(U) \subset S \to \mathbb{R}^2$ is differentiable and we conclude*

$$\begin{cases} \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbf{x}(U) \subset S \text{ is one-one, onto, differentiable} \\ \mathbf{x}^{-1} : \mathbf{x}(U) \subset S \to U \subset \mathbb{R}^2 \text{ is one-one, onto, differentiable,} \end{cases}$$

which implies that the two regular surfaces $U \subset \mathbb{R}^2$ ***(note that $U \subset \mathbb{R}^2$ can be viewed as a regular surface in $\mathbb{R}^3$)*** *and $\mathbf{x}(U) \subset S$ are* ***diffeomorphic.*** *We conclude that* ***a regular surface S is a subset in $\mathbb{R}^3$ such that it is locally diffeomorphic*** *to an open subset of the plane $\mathbb{R}^2$.*

Example 2.54 *(This is Example 3 in p. 77.) Let* S_1 *and* S_2 *be two regular surfaces such that* $S_1 \subset V$ *(open set in* $\mathbb{R}^3$*)* $\subset \mathbb{R}^3$*, and that* $\varphi : V \to \mathbb{R}^3$ *is a differentiable map with* $\varphi(S_1) \subset S_2$*. Then* ***the restriction*** $\varphi|_{S_1} : S_1 \to S_2$ *is a* ***differentiable map****. To see this, we look at the map*

$$\mathbf{x}_2^{-1} \circ \underbrace{\varphi \circ \mathbf{x}_1} : U_1 \to U_2,$$

where the map $\varphi \circ \mathbf{x}_1$ *is a differentiable map from* U_1 *(open set in* $\mathbb{R}^2$*) to* S_1 *and by Theorem 2.46 (see (123)), we know that* $\mathbf{x}_2^{-1} \circ \underbrace{\varphi \circ \mathbf{x}_1}$ *is a differentiable map on* U_1*. Continue with 1, 2, 3 in P. 77.*

Before, we consider "parametrized curves" in $\mathbb{R}^3$. Now, similar to the definition of regular surfaces in $\mathbb{R}^3$, one can also consider "**regular curves**" in $\mathbb{R}^3$. Its definition is similar to that of a regular surface.

Definition 2.55 *A* ***regular curve*** C *in* $\mathbb{R}^3$ *is a subset of* $\mathbb{R}^3$ *with the property that for each* $p \in C$ *there is a neighborhood* V *of* p *in* $\mathbb{R}^3$ *and a differentiable homeomorphism* $\alpha : I \subset \mathbb{R} \to V \bigcap C$ *(here* $I \subset \mathbb{R}$ *is an open interval) such that the differential* $d\alpha_t$ *is one-one (i.e., the linear map* $d\alpha_t$ *has rank one) for each* $t \in I$*. The map* α *is called a* ***local parametrization*** *of* C*.*

Remark 2.56 *Unlike before (in Chapter 1), here a regular curve* C *has* ***no self-intersections****.*

Remark 2.57 *A change of parameters for a regular curve* C *is a* ***diffeomorphism*** *from some open interval of* $\mathbb{R}$ *to another open interval of* $\mathbb{R}$*.*

Example 2.58 *(****Surface of revolution by a generating curve*** C*.) Let* C *be a* ***regular connected plane curve*** *(according to Definition 2.55) lying on* xz*-plane parametrized by*

$$(x, 0, z) = (f(v), 0, g(v)), \quad v \in (a, b),$$

where $(f'(v), 0, g'(v)) \neq (0, 0, 0)$ *and* $f(v) > 0$ *for all* $v \in (a, b)$*. Here the map* $v \to (f(v), 0, g(v))$ *is assumed to be* ***one-one*** *and onto between* (a, b) *and* C*. Let* $S \subset \mathbb{R}^3$ *be the set obtained by rotating* C *with respect to* z*-axis. We claim that* S *is a regular surface in* $\mathbb{R}^3$*, called a* ***surface of revolution****. To see this, consider the map*

$$\mathbf{x}(u, v) = (f(v)\cos u,\ f(v)\sin u,\ g(v)), \quad (u, v) \in (0, 2\pi) \times (a, b).$$

Then $\mathbf{x} : U = (0, 2\pi) \times (a, b) \to S \subset \mathbb{R}^3$ *is differentiable with rank* 2 *everywhere. If we have*

$$\mathbf{x}(u_0, v_0) = \mathbf{x}(u_1, v_1) \quad \text{for} \quad (u_0, v_0) \neq (u_1, v_1) \in U,$$

then $f^2(v_0) = f^2(v_1)$ *(which implies* $f(v_0) = f(v_1)$ *since* $f(v) > 0$ *for all* $v \in (a, b)$*),* $g(v_0) = g(v_1)$*. Hence we have* $v_0 = v_1$ *and then* $u_0 = u_1$*. We claim that* $\mathbf{x} : U \to \mathbf{x}(U) = S - \{C\}$ *is a homeomorphism. We first note that since* $(a, b) \to C$ *is a homeomorphism,* v *is a continuous function of* $z = g(v)$ *and* $\sqrt{x^2 + y^2} = f(v)$*. Hence* v *is a continuous function of* $(x, y, z) \in S - \{C\}$*. For* $u \in (0, 2\pi)$*, it can be expressed as*

$$u = \begin{cases} \cot^{-1}\frac{x}{y}, & \text{if} \quad u \in (0, \pi) \ \ (y \neq 0 \text{ on this interval}) \\ \pi + \cot^{-1}\frac{x}{y}, & \text{if} \quad u \in (\pi, 2\pi) \ \ (y \neq 0 \text{ on this interval}) \\ \pi + \tan^{-1}\frac{y}{x}, & \text{if} \quad u \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \ \ (x \neq 0 \text{ on this interval}), \end{cases}$$

where we note that, by definition, $\cot^{-1}\frac{x}{y} \in (0, \pi)$ *and* $\tan^{-1}\frac{y}{x} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$*. Hence* $(u, v) \in U$ *is a continuous function of* $(x, y, z) \in S - \{C\}$ *(you can also see the argument in the textbook). Finally we see that* S *can be covered by two such parametrizations. The circles generated by points of* C *are called* ***parallels*** *of* S *and curves on* S *coming from a rotation of* C *are called* ***meridians*** *of* S*.*

Similar to the definition of a parametrized differentiable curve in Chapter 1, we can also define the following:

Definition 2.59 *A* ***parametrized surface*** *is a* ***differentiable*** *map* $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *from some open set* $U \subset \mathbb{R}^2$ *into* $\mathbb{R}^3$*. The set* $\mathbf{x}(U)$ *is called the* ***trace*** *of* $\mathbf{x}$ *and we say* $\mathbf{x}$ *is* ***regular*** *if the differential* $d\mathbf{x}_q$ *has rank 2 for* ***all*** $q \in U$*. If* $d\mathbf{x}_q$ *has rank 2, we say* $q \in U$ *is a* ***regular point*** *of* $\mathbf{x}$*. If* $d\mathbf{x}_q$ *does not have rank 2, we say* $q \in U$ *is a* ***singular point*** *of* $\mathbf{x}$*.*

Remark 2.60 *A regular surface (in the definition in p. 54 of the textbook) is locally a regular parametrized surface.*

Remark 2.61 *In general, a* ***parametrized surface****, even regular everywhere, may have self-intersections.*

Remark 2.62 *So until now, we have* ***regular parametrized curves*** *in* $\mathbb{R}^3$ *and* ***regular curves*** *in* $\mathbb{R}^3$*. We also have* ***regular parametrized surfaces*** *in* $\mathbb{R}^3$ *and* ***regular surfaces*** *in* $\mathbb{R}^3$*.*

Example 2.63 *(This is Example 5 in p. 81.) Let* $I = (a, b) \subset \mathbb{R}$ *be an open interval and* $\alpha : I \to \mathbb{R}^3$ *be a* ***nonplanar*** *regular parametrized curve and define*

$$\mathbf{x}(t, v) = \alpha(t) + v\alpha'(t), \quad (t, v) \in I \times \mathbb{R}.$$

Then $\mathbf{x}(t, v)$ *is a parametrized surface called the* ***tangent surface*** *of* α*. Assume that the curvature* $k(t)$ *of* $\alpha(t)$ *is nonzero for all* $t \in I$ *and let* $U = \{(t, v) : t \in I,\ v \neq 0\} = I \times \mathbb{R} \backslash \{0\}$*. Note that* $U \subset \mathbb{R}^2$ *is not connected. Then* $\mathbf{x} : U \to \mathbf{x}(U)$ *has rank 2 due to*

$$\frac{\partial \mathbf{x}}{\partial t} \wedge \frac{\partial \mathbf{x}}{\partial v} = v\alpha''(t) \wedge \alpha'(t) \neq 0, \quad \forall\ (t, v) \in U,$$

since

$$k(t) = \frac{|\alpha'(t) \wedge \alpha''(t)|}{|\alpha'(t)|^3}, \quad t \in I.$$

Thus $\mathbf{x} : U \to \mathbf{x}(U)$ *is a* ***regular parametrized surface****. Its trace consists of two connected pieces whose common boundary is the set* $\alpha(I)$*.*

The relation between regular parametrized surfaces and regular surfaces is the following:

Theorem 2.64 *Let* $U \subset \mathbb{R}^2$ *be an open set and* $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *be a regular parametrized surface and* $q \in U$*. Then there exists a neighborhood* V *of* q *in* $\mathbb{R}^2$*,* $V \subset U$*, such that* $\mathbf{x}(V) \subset \mathbb{R}^3$ *is a* ***regular surface*** *(in the definition in p. 54 of the textbook). That is, a regular parametrized surface is locally a regular surface.*

Remark 2.65 *(****Be careful.****) In the above theorem the neighborhood* $V \subset \mathbb{R}^2$ *is taken in the domain space* $\mathbb{R}^2$*, not in the target space* $\mathbb{R}^3$*. This is because* $\mathbf{x}(U)$ *may have self-intersections and* $\mathbf{x}(q)$ *may happen to be an intersection point.*

Proof. (My proof is slightly different from the book proof.) Again, we use the inverse function theorem. Write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

and by regularity ($d\mathbf{x}_p$ has rank 2 for all points $p \in U$), without loss of generality, we may assume that $(\partial(x, y)/\partial(u, v))(q) \neq 0$, i.e.

$$\frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0, \quad q = (u_0, v_0) \in U.$$

Denote $(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0)) = (x_0,y_0,z_0)$. By the **inverse function theorem** applied to the map

$$(u,v)\ \text{(near } (u_0,v_0)\text{)} \to \pi\circ\mathbf{x}(u,v) = (x(u,v), y(u,v))\ \text{(near } (x_0,y_0)\text{)},$$

one can solve (u,v) in terms of (x,y) near (x_0,y_0) and near $q=(u_0,v_0)$, the map $\mathbf{x}(u,v)$ when in terms of (x,y) has the form

$$\mathbf{x}(u,v) = \mathbf{x}(u(x,y), v(x,y)) = (x,\ y,\ z(u(x,y), v(x,y))),\quad \forall\ (u,v)\in V,$$

where V is some small neighborhood of (u_0,v_0) in U. Hence we see that $\mathbf{x}(V)\subset\mathbb{R}^3$ is the **graph** of a differentiable function $z=f(x,y)$ and is a **regular surface**. □

2.4 The Tangent Plane; the Differential of a Map (this is Section 2.4 of the textbook).

Definition 2.66 *Let $p\in S$. If there exists a differentiable parametrized curve $\alpha(t):(-\varepsilon,\varepsilon)\to S$ with $\alpha(0)=p$ and $\alpha'(0)=v\in\mathbb{R}^3$, then we say v is a* ***tangent vector*** *to S at the point p.*

Definition 2.67 *The set of all tangent vectors to S at $p\in S$ is called* ***the tangent space*** *of S at p. We denote it as T_pS.*

We next review some elementary facts from advanced calculus. Let $f:U\subset\mathbb{R}^n\to\mathbb{R}$ be a differentiable function and $\beta(t):(-\varepsilon,\varepsilon)\to U$ be a differentiable curve in U with $\beta(0)=p\in U$, $\beta'(0)=v\in\mathbb{R}^n$. Then $\alpha(t):=f(\beta(t)):(-\varepsilon,\varepsilon)\to\mathbb{R}$ is also a differentiable function with $\alpha(0)=f(p)$ and by the chain rule we know that

$$\alpha'(0) = \langle\nabla f(\beta(0)), \beta'(0)\rangle = \langle\nabla f(p), v\rangle, \tag{125}$$

where

$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p),\ \frac{\partial f}{\partial x_2}(p),\ \dots,\ \frac{\partial f}{\partial x_n}(p)\right),\quad p\in U.$$

The value of $\alpha'(0)$ **does not** depend on the curve $\beta(t)$ as long as it satisfies $\beta(0)=p$ and $\beta'(0)=v$.

Now if $f=(f_1,\dots,f_m):U\subset\mathbb{R}^n\to\mathbb{R}^m$ is a differentiable function and $\beta(t):(-\varepsilon,\varepsilon)\to U$ is a differentiable curve in U with $\beta(0)=p\in U$, $\beta'(0)=v\in\mathbb{R}^n$. The composition $\alpha(t):=f(\beta(t))=(f_1(\beta(t)),\dots,f_m(\beta(t))):(-\varepsilon,\varepsilon)\to\mathbb{R}^m$ is a curve in $\mathbb{R}^m$ with $\alpha(0)=f(p)$ and the **chain rule** implies

$$\begin{aligned}\alpha'(t)|_{t=0} &= \left(\frac{d}{dt}f_1(\beta(t))\Big|_{t=0},\ \dots,\ \frac{d}{dt}f_m(\beta(t))\Big|_{t=0}\right)\\ &= (\langle\nabla f_1(p), v\rangle,\ \dots,\ \langle\nabla f_m(p), v\rangle) = df_p(v),\end{aligned} \tag{126}$$

where $df_p(\cdot):\mathbb{R}^n\to\mathbb{R}^m$ is the **differential** (or **derivative** or **total derivative**) of f at p. From advanced calculus, we know that $df_p(\cdot):\mathbb{R}^n\to\mathbb{R}^m$ is a linear map.

Again, the vector $\alpha'(t)|_{t=0}$ **does not depend on** $\beta(t)$ as long as it satisfies $\beta(0)=p$ and $\beta'(0)=v$. For two different $\beta(t)$, $\tilde\beta(t)$ with $\beta(0)=\tilde\beta(0)=p$ and $\beta'(0)=\tilde\beta'(0)=v$, the two curves $\alpha(t)=f(\beta(t))$ and $\tilde\alpha(t)=f\left(\tilde\beta(t)\right)$ in $\mathbb{R}^m$ both pass the point $f(p)\in\mathbb{R}^m$ and have **the same tangent vector** $df_p(v)$.

Remark 2.68 *Draw a picture for the above discussion.*

Important observation: From the above observation, to compute $df_p(v)$, one can choose a convenient $\beta(t)$ with $\beta(0)=p$, $\beta'(0)=v$ and evaluate $\frac{d}{dt}f(\beta(t))\big|_{t=0}$.

We also have the following interesting observation:

Lemma 2.69 *Let $f,\ g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be two differentiable maps such that $f = g$ along two differentiable curves $\alpha(t),\ \beta(t) \in U$ with $\alpha(0) = \beta(0) = p \in U$ and $\alpha'(0)$ is **independent** to $\beta'(0)$. Then we have $df_p(v) = dg_p(v)$ for **all** $v \in \mathbb{R}^2$.*

Proof. Exercise. □

The following says that the tangent space of a regular surface S at $p \in S$ is a vector space of dimension 2.

Theorem 2.70 *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a regular surface S and $q \in U$. The vector space of dimension 2, given by*

$$d\mathbf{x}_q\left(\mathbb{R}^2\right) \subset \mathbb{R}^3,$$

*coincides with the set of all tangent vectors to S at the point $p = \mathbf{x}(q)$, i.e., $d\mathbf{x}_q(\mathbb{R}^2) = T_pS$. In particular, we see that T_pS is a **plane** in $\mathbb{R}^3$ passing through p and the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to T_pS$ is one-one and onto.*

Remark 2.71 *By the above theorem, the plane $d\mathbf{x}_q(\mathbb{R}^2)$, which passes through $p = \mathbf{x}(q)$ does not depend on the parametrization $\mathbf{x}$. This is because our definition of T_pS does not involve any parametrization.*

Proof. Let $w \in T_pS$. Then there exists a differentiable map $\alpha(t) : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = w$. Now the curve $\beta(t) := \mathbf{x}^{-1}(\alpha(t)) : (-\varepsilon, \varepsilon) \to U \subset \mathbb{R}^2$ is differentiable (see Theorem 2.46) with $\beta(0) = q$. By the chain rule, we have

$$\begin{aligned} d\mathbf{x}_q(\beta'(0)) &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{x}(\beta(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{x}\left(\mathbf{x}^{-1}(\alpha(t))\right) = \left.\frac{d}{dt}\right|_{t=0} \alpha(t) = w, \quad \beta'(0) \in \mathbb{R}^2, \end{aligned}$$

which implies $w \in d\mathbf{x}_q(\mathbb{R}^2)$ and so $T_pS \subset d\mathbf{x}_q(\mathbb{R}^2)$

Conversely, let $w = d\mathbf{x}_q(v)$ for some $v \in \mathbb{R}^2$. Then v is the velocity vector of the curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ given by $\gamma(t) = q + tv,\ \gamma(0) = q \in U$. Now we look at the differentiable curve $\alpha = \mathbf{x} \circ \gamma : (-\varepsilon, \varepsilon) \to S$. By the chain rule, we have

$$\alpha'(0) = d\mathbf{x}_q(\gamma'(0)) = d\mathbf{x}_q(v) = w \in T_pS.$$

Hence we have $d\mathbf{x}_q(\mathbb{R}^2) \subset T_pS$. The proof is done. □

Recall that if $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable map (here U is an open set in $\mathbb{R}^3$) and $a \in f(U)$ is a **regular value**, then $f^{-1}(a) \subset U$ is a **regular surface** in $\mathbb{R}^3$. For any $p \in S = f^{-1}(a)$, the tangent space T_pS can be described as the following:

Lemma 2.72 *Let $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a differentiable map and $a \in f(U)$ is a **regular value**, then*

$$T_pS = \ker\left\{df_p : \mathbb{R}^3 \to \mathbb{R}\right\}, \quad p \in S = f^{-1}(a). \tag{127}$$

Proof. Since a is a **regular value** of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, the map $df_p : \mathbb{R}^3 \to \mathbb{R}$ is onto and the vector space $\ker\{df_p : \mathbb{R}^3 \to \mathbb{R}\}$ has dimension 2. For any curve $\alpha(t) \in S = f^{-1}(a)$ with $\alpha(0) = p,\ \alpha'(0) = v \in T_pS$, we have $f(\alpha(t)) = a$ for all $t \in (-\varepsilon, \varepsilon)$. Hence $df_p(v) = (f \circ \alpha)'(0) = 0$ and so $T_pS \subset \ker\{df_p : \mathbb{R}^3 \to \mathbb{R}\}$. Since both spaces have dimension 2, we conclude that $T_pS = \ker\{df_p : \mathbb{R}^3 \to \mathbb{R}\}$. □

Let $\mathbf{x}: U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a parametrization of a regular surface S and $q \in U$. The choice of the parametrization $\mathbf{x}$ determines a basis

$$\{d\mathbf{x}_q(e_1),\ d\mathbf{x}_q(e_2)\} = \left\{\frac{\partial \mathbf{x}}{\partial u}(q),\ \frac{\partial \mathbf{x}}{\partial v}(q)\right\} \quad \text{(also denote it as } \{\mathbf{x}_u(q),\ \mathbf{x}_v(q)\}\text{)}$$

of T_pS. We call it **the basis associated to the parametrization** $\mathbf{x}$. If $(u(t), v(t)) \in U$ is a curve with $(u(0), v(0)) = q$ and $(u'(0), v'(0)) = w$, then the vector $d\mathbf{x}_q(w)$ is given by

$$d\mathbf{x}_q(w) = u'(0)\,\mathbf{x}_u(q) + v'(0)\,\mathbf{x}_v(q). \tag{128}$$

Definition 2.73 *(**The differential of a function defined on regular surface** $S \subset \mathbb{R}^3$.) Let $\varphi: V \subset S \to \mathbb{R}$ (or $\mathbb{R}^n$) be a differentiable function, where $V \subset S$ is an open set and $p \in V$. For any $v \in T_pS$ one can choose a curve $\alpha(t) \in V$, $t \in (-\varepsilon, \varepsilon)$, with $\alpha(0) = p$ and $\alpha'(0) = w$. Then $\beta(t) = \varphi(\alpha(t))$, $t \in (-\varepsilon, \varepsilon)$, is a real-valued function. We **define** $d\varphi_p(w) = \beta'(0)$, where $d\varphi_p: T_pS \to \mathbb{R}$ (or $\mathbb{R}^n$) is called the **differential** of φ at $p \in V$.*

Lemma 2.74 *In the above definition, the quantity $d\varphi_p(w) \in \mathbb{R}$ (or $\mathbb{R}^n$) is **independent of** the choice of the curve $\alpha(t) \in V$ as long as $\alpha(0) = p$ and $\alpha'(0) = w$.*

Proof. Let $\mathbf{x}: U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a local parametrization around $p \in S$ with $\mathbf{x}(q) = p$, $q \in U$. Let $\gamma(t)$ be another curve in V with $\gamma(0) = p$ and $\gamma'(0) = w$, and let $\tilde{\alpha}(t) = \mathbf{x}^{-1}(\alpha(t))$, $\tilde{\gamma}(t) = \mathbf{x}^{-1}(\gamma(t))$ be the corresponding curves in U. We have $\tilde{\alpha}(0) = \tilde{\gamma}(0) = q \in U$. We also know that (note that the domain of $\mathbf{x}$ is an open set in $\mathbb{R}^2$ and so the familiar chain rule in advanced calculus is valid here)

$$d\mathbf{x}_q(\tilde{\alpha}'(0)) = \frac{d}{dt}\mathbf{x}(\tilde{\alpha}(t))\Big|_{t=0} = \frac{d}{dt}\alpha(t)\Big|_{t=0} = w$$

and

$$d\mathbf{x}_q(\tilde{\gamma}'(0)) = \frac{d}{dt}\mathbf{x}(\tilde{\gamma}(t))\Big|_{t=0} = \frac{d}{dt}\gamma(t)\Big|_{t=0} = w.$$

Since $d\mathbf{x}_q: \mathbb{R}^2 \to T_pS$ $(= d\mathbf{x}_q(\mathbb{R}^2))$ is an **isomorphism** (both $\mathbb{R}^2$ and T_pS have the same dimension), we must have

$$\tilde{\alpha}'(0) = \tilde{\gamma}'(0), \qquad \tilde{\alpha}(t) \in U, \qquad \tilde{\gamma}(t) \in U, \qquad t \in (-\varepsilon, \varepsilon).$$

Now the map $\varphi \circ \mathbf{x}: U \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable (we may assume $\mathbf{x}(U) \subset V$). Hence we know that

$$\frac{d}{dt}(\varphi \circ \mathbf{x})(\tilde{\alpha}(t))\Big|_{t=0} = \frac{d}{dt}(\varphi \circ \mathbf{x})(\tilde{\gamma}(t))\Big|_{t=0},$$

which implies

$$\frac{d}{dt}\varphi(\alpha(t))\Big|_{t=0} = \frac{d}{dt}\varphi(\gamma(t))\Big|_{t=0}.$$

Thus the definition of the value $d\varphi_p(w)$ is independent of the choice of the curve $\alpha(t)$. □

Remark 2.75 *(**Important.**) From the above definition, the differentiable map $\mathbf{x}^{-1}: \mathbf{x}(U) \subset S \to \mathbb{R}^2$ has differential at $p = \mathbf{x}(q)$, i.e. $d(\mathbf{x}^{-1})_p: T_pS \to \mathbb{R}^2$. One can check that for any $v \in T_pS$ we have:*

$$d\mathbf{x}_q \circ d(\mathbf{x}^{-1})_p(v) = d\mathbf{x}_q\left(\frac{d}{dt}\Big|_{t=0}\mathbf{x}^{-1}(\alpha(t))\right) \quad \textit{(here } \alpha(0) = p,\ \alpha'(0) = v \in T_pS\textit{)}$$
$$\overset{chain\ rule}{=} \frac{d}{dt}\Big|_{t=0}\left(\mathbf{x} \circ \mathbf{x}^{-1}(\alpha(t))\right) = \frac{d}{dt}\Big|_{t=0}\alpha(t) = v, \quad \forall\ v \in T_pS.$$

*Since $d\mathbf{x}_q: \mathbb{R}^2 \to T_pS$ is a linear isomorphism, the map $d(\mathbf{x}^{-1})_p: T_pS \to \mathbb{R}^2$ is the **inverse** of the linear map $d\mathbf{x}_q: \mathbb{R}^2 \to T_pS$, where $\mathbf{x}(q) = p \in S$. Note that by Lemma 2.74, the vector $d(\mathbf{x}^{-1})_p(v)$ is independent of the choice of the curve $\alpha(t)$ as long as it satisfies $\alpha(0) = p \in S$ and $\alpha'(0) = v \in T_pS$.*

Definition 2.76 *(**The differential of a function defined on regular surface** $S \subset \mathbb{R}^3$.) Let $\varphi : V \subset S_1 \to S_2$ be a differentiable map, where $V \subset S_1$ is an open set and $p \in V$. For any $v \in T_pS_1$ one can choose a curve $\alpha(t) \in V,\ t \in (-\varepsilon, \varepsilon)$, with $\alpha(0) = p$ and $\alpha'(0) = w$. Then the curve $\beta(t) = \varphi(\alpha(t)),\ t \in (-\varepsilon, \varepsilon)$, will be a curve in S_2 passing through $\varphi(p)$. We define*

$$d\varphi_p(w) = \beta'(0) \in T_{\varphi(p)}S_2$$

and the map

$$d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$$

*is called the **differential** of φ at $p \in V$.*

The following says that the above definition is well-defined.

Theorem 2.77 *The vector $d\varphi_p(w)$ is **independent of** the choice of the curve $\alpha(t)$ (as long as it satisfies $\alpha(0) = p$ and $\alpha'(0) = w$) and the differential $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ is a **linear map** between two tangent spaces.*

Proof. (**Read it by yourself.**) See Remark 2.75 first.

Let $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ be parametrizations in neighborhoods of p and $\varphi(p)$ respectively and suppose that φ is expressed (at the **bottom level**) in these coordinates by

$$\varphi(u, v) = (\varphi_1(u, v),\ \varphi_2(u, v))$$

and α is expressed by $\alpha(t) = (u(t), v(t)),\ t \in (-\varepsilon, \varepsilon)$. Then we have

$$\beta(t) = (\varphi_1(u(t), v(t)),\ \varphi_2(u(t), v(t)))$$

and by the chain rule

$$\beta'(0) = \left(\frac{\partial \varphi_1}{\partial u}(*)\, u'(0) + \frac{\partial \varphi_1}{\partial v}(*)\, v'(0),\ \frac{\partial \varphi_2}{\partial u}(*)\, u'(0) + \frac{\partial \varphi_2}{\partial v}(*)\, v'(0)\right)$$

where $(*) = (u(0), v(0))$. Thus $\beta'(0)$ depends only on the map φ and the coordinates $(u'(0), v'(0))$ of the vector w is the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$. In particular, it is independent of the curve $\alpha(t)$ (as long as $\alpha(0) = p$ and $\alpha'(0) = w$).

We can see that

$$\beta'(0) = d\varphi_p(w) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial u}(*) & \frac{\partial \varphi_1}{\partial v}(*) \\ \frac{\partial \varphi_2}{\partial u}(*) & \frac{\partial \varphi_2}{\partial v}(*) \end{pmatrix} \begin{pmatrix} u'(0) \\ v'(0) \end{pmatrix}. \tag{129}$$

By Remark 2.75, this implies that $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ (at the **top level**) is **linear** and its matrix representation with respect to the bases $\{\mathbf{x}_u, \mathbf{x}_v\}$ of T_pS_1 and $\{\bar{\mathbf{x}}_{\bar{u}}, \bar{\mathbf{x}}_{\bar{v}}\}$ of $T_{\varphi(p)}S_2$ is the matrix given by (129). □

Example 2.78 *Read Example 1 and Example 2 in p. 88 by yourself ...*

The most important for differentiable maps between regular surfaces is probably the following: The chain rule in Euclidean spaces can be generalized to surfaces.

Lemma 2.79 *(**Chain rule for differentiable maps on regular surfaces**.) (This is Exercise 24 in p. 93.) Let $S_1,\ S_2,\ S_3$ be three regular surfaces in $\mathbb{R}^3$ and let $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$ be two differentiable maps. For any $p \in S_1$ we have*

$$d(\psi \circ \varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p \tag{130}$$

where we note that both sides of (130) are linear maps from T_pS_1 to $T_{\psi(\varphi(p))}S_3$.

Proof. For any $v \in T_pS_1$, choose $\alpha(t) \in S_1$ with $\alpha(0) = p,\ \alpha'(0) = v$. Then

$$d\psi_{\varphi(p)} \circ d\varphi_p(v) = d\psi_{\varphi(p)}\left(\left.\frac{d}{dt}\right|_{t=0} \varphi(\alpha(t))\right) = \left.\frac{d}{dt}\right|_{t=0} \psi(\varphi(\alpha(t))) = d(\psi \circ \varphi)_p(v).$$

The proof is done. □

Remark 2.80 *By the above we see that if $\varphi : S_1 \to S_2$ is a diffeomorphism, then*

$$(d\varphi_p)^{-1} = d(\varphi^{-1})_{\varphi(p)}. \tag{131}$$

Definition 2.81 *A mapping $\varphi : U \subset S_1 \to S_2$ between two regular surfaces in $\mathbb{R}^3$ is called a **local diffeomorphism** at $p \in U$ if there exists a neighborhood $V \subset U$ of p such that $\varphi : V \to \varphi(V)$ is a diffeomorphism.*

We can extend the important inverse function theorem in calculus to differentiable mappings between regular surfaces.

Theorem 2.82 *(**Inverse function theorem on surfaces**.) (This is Proposition 3 in p. 89.) Let $S_1,\ S_2$ be two regular surfaces in $\mathbb{R}^3$. If $\varphi : U \subset S_1 \to S_2$ is a differentiable map such that the differential $d\varphi_p : T_pS_1 \to T_{\varphi(p)}S_2$ is an isomorphism at $p \in U$, then φ is a **local diffeomorphism** at p.*

Proof. Let $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ be parametrizations in neighborhoods of p and $\varphi(p)$ respectively. The map $\psi = \bar{\mathbf{x}}^{-1} \circ \varphi \circ \mathbf{x} : O \subset \mathbb{R}^2 \to \mathbb{R}^2$ is defined on some open set O in $\mathbb{R}^2$ containing $q = \mathbf{x}^{-1}(p)$ and, by Remark 2.75 and Lemma 2.79, we see that $d\psi_q : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism. Hence by the classical inverse function theorem, there exist open set $O_1 \subset O \subset \mathbb{R}^2$ and open set $O_2 \subset \mathbb{R}^2$ such that $\psi : O_1 \to O_2$ is a diffeomorphism. Now $\varphi = \bar{\mathbf{x}} \circ \psi \circ \mathbf{x}^{-1} : \mathbf{x}(O_1) \to \bar{\mathbf{x}}(O_2)$ is a diffeomorphism and the proof is done. □

Example 2.83 *(**Interesting**.) Let $S \subset \mathbb{R}^3$ be a **compact regular surface** (this means that $S \subset \mathbb{R}^3$ is a regular surface and is also a **compact set** in $\mathbb{R}^3$, for example, a sphere or an ellipsoid) and $\varphi : S \to \mathbb{R}^2$ is a differentiable map (note that both S and $\mathbb{R}^2$ have the same dimension). We claim that the differential $d\varphi_p : T_pS \to \mathbb{R}^2$ **cannot be nonsingular** for all $p \in S$ (i.e. $\varphi : S \to \mathbb{R}^2$ **must have critical point on** S). To see this, since the set $\varphi(S)$ is **compact** in $\mathbb{R}^2$, there exists a point $p_0 \in S$ such that $|\varphi(p_0)| > 0$ is the **maximum** of φ on S. If $d\varphi_{p_0} : T_{p_0}S \to \mathbb{R}^2$ is nonsingular, then by the above **inverse function theorem**, there exist neighborhood of p_0 in S and neighborhood of $\varphi(p_0)$ in $\mathbb{R}^2$ such that φ on these two neighborhoods is a **diffeomorphism**. This will imply that $|\varphi(p_0)| > 0$ cannot be the maximum of φ on S.*

Lemma 2.84 *(**This is Exercise 21 in book p. 93.**) Let S be a **connected** regular surface. If $f : S \to \mathbb{R}$ (or $\mathbb{R}^n$) is differentiable with $df_p = 0$ for all $p \in S$, then f is a constant map.*

Remark 2.85 *The terminology "connected" in this book is defined in p. 63 of the book, which actually means "**path connected**". There is a property in general topological space which says that **"path connected" implies "connected"** (which in topology means that S cannot be decomposed as $S = A \bigcup B$, where both A and B are nonempty open sets in S. In particular, if a set $A \subset S$ is both open and closed in S, then either $A = \varnothing$ or $A = S$.*

Remark 2.86 *In advanced calculus there is the following result: Let $U \subset \mathbb{R}^n$ be an **open connected set** and $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable map. If we have $dF_p = 0$ for all $p \in U$, then F must be a constant map. We will use this property in the following proof.*

Proof. See Remark 2.85 first.

Choose $a \in f(S)$ and consider the set

$$A = \{p \in S : f(p) = a\} \subset S.$$

The set $A \neq \varnothing$ since a is in the **image** of f and there exists some $p_0 \in S$ with $f(p_0) = a$, i.e. $p_0 \in A$. Clearly A is closed in S. We claim that A is also open in S. For any $p \in A \subset S$, there exists a parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbf{x}(U) \subset S$ satisfying: U is open and connected in $\mathbb{R}^2$; $\mathbf{x}(U)$ is open and connected in S; $p \in \mathbf{x}(U)$. The map $f \circ \mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^n$) will satisfy (by chain rule and the assumption)

$$d(f \circ \mathbf{x})_q = df_{\mathbf{x}(q)} \circ d\mathbf{x}_q = 0, \quad \forall\, q \in U.$$

Since U is open and connected in $\mathbb{R}^2$, result in advanced calculus says that $f \circ \mathbf{x}$ must be a **constant function** on U. Therefore, f is a constant function C on the open set $\mathbf{x}(U)$ and, by $p \in \mathbf{x}(U)$, the constant C must be equal to $f(p)$, i.e. $C = f(p) = a$. Hence the whole open set $\mathbf{x}(U) \subset A$ and A is closed in S. Now we conclude $A = S$ and so f is a constant map on S. □

Definition 2.87 *Let $S \subset \mathbb{R}^3$ be a regular surface. If $f : S \to \mathbb{R}$ is differentiable with $df_p = 0$ on T_pS, then we say $p \in S$ is a* ***critical point*** *of f on S.*

Remark 2.88 *Similar to Definition 2.16, we say $p \in S$ is a* ***critical point*** *of f on S if the map $df_p : T_pS \to \mathbb{R}$ is not onto. Since $\dim \mathbb{R} = 1$, we see not "not onto" is equivalent to "$df_p = 0$ on T_pS".*

The following is obvious:

Lemma 2.89 *If $f : S \to \mathbb{R}$ is differentiable and $p \in S$ is a local* ***maximum*** *(or* ***minimum****) of f, then p is a critical point of f on S.*

Proof. Choose $\alpha(t) : (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$, $\alpha'(0) = v \in T_pS$. Since $\alpha(0) = p$ is a local **maximum** (or **minimum**) of f, we have

$$df_p(v) = \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t)) = 0.$$

Since $\alpha'(0) = v \in T_pS$ is arbitrary, we have $df_p(v) = 0$ for all $v \in T_pS$ and p is a critical point of f on S. □

Definition 2.90 *Let $\mathbf{x} : U \subset \mathbb{R}^2 \to S$ be a parametrization of a regular surface. The vector*

$$N(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q) \in \mathbb{R}^3, \quad q \in U \tag{132}$$

is called the ***unit normal vector field*** *on $\mathbf{x}(U) \subset S$ induced by the parametrization $\mathbf{x}$. Note: $N(q)$ is normal to S at the point $p = \mathbf{x}(q)$.*

Lemma 2.91 *The map $N : \mathbf{x}(U) \subset S \to \mathbb{R}^3$ is* ***differentiable*** *on $\mathbf{x}(U)$.*

Proof. This is obvious if we use local coordinate $\mathbf{x}$ to express N. We have

$$N \circ \mathbf{x}(q) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}\left(\mathbf{x}^{-1}(\mathbf{x}(q))\right) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q), \quad q \in U.$$

Since the denominator is not zero for all $q \in U$, $\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(q)$ is a differentiable function on U. The result follows. □

Definition 2.92 *Let $p \in S$. The line passing through p and is perpendicular to T_pS is called the **normal line** of S at p.*

Example 2.93 *(**Simple but interesting.**) Let $p_0 \in \mathbb{R}^3$ with $p_0 \notin S$, and $f : S \to \mathbb{R}$ is the square of the distance (in $\mathbb{R}^3$) to p_0 given by*

$$f(p) = |p - p_0|^2, \quad p \in S. \tag{133}$$

For any $v \in T_pS$, we have

$$df_p(v) = \left.\frac{d}{dt}\right|_{t=0} |\alpha(t) - p_0|^2 = 2\langle v,\ p - p_0\rangle, \quad \alpha(0) = p, \quad \alpha'(0) = v.$$

*Hence if $p \in S$ is a **critical point** of f if and only if the line segment $\overline{p_0p}$ is **normal** to S at p. Moreover, if S is a **compact regular surface** in $\mathbb{R}^3$ (this means that $S \subset \mathbb{R}^3$ is a regular surface and is also a **compact set** in $\mathbb{R}^3$, for example, a sphere or an ellipsoid), then $f : S \to \mathbb{R}$ has at least one critical point $p \in S$ (because f on S has **maximum** value and **minimum** value attained on S). From this, we conclude:*

> *From any point $p_0 \in \mathbb{R}^3$, $p_0 \notin S$, we can draw a normal line to a given **compact surface** S.*

*If S is **connected** (not necessarily compact) and $df_p = 0$ **for all** $p \in S$ (i.e., all points on S are critical points of f), then f must be a **constant map**, i.e.*

$$f(p) = |p - p_0|^2 = C > 0, \quad \forall\ p \in S. \tag{134}$$

From this, we conclude:

> *If all normal lines of a connected surface S meet at a given point $p_0 \in \mathbb{R}^3$, then the surface is contained in a sphere centered at p_0.*

*A different way to say the above is that if $p_0 \notin S$ is such that the line segment $\overline{p_0p}$ is **normal** to S **for all** $p \in S$, then S must be contained in a sphere centered at p_0.*

2.4.1 Diagonalization of a 3×3 symmetric matrix via critical point theory.

Remark 2.94 ***We can use calculus to solve an algebra problem ... The method can be generalized to the case** $A \in M(n)$, **where** A **is symmetric.***

One can use critical point theory discussed in the above to diagonalize a 3×3 symmetric matrix. Let $A \in M(3)$ be symmetric. From linear algebra, we know that it has 3 **real** eigenvalues.

Consider the differentiable map $f : S^2 \to \mathbb{R}$ given by

$$f(p) = \langle Ap,\ p\rangle, \quad p \in S^2, \tag{135}$$

where $S^2 \subset \mathbb{R}^3$ is the unit sphere (a regular surface) in $\mathbb{R}^3$ given by $S^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^3$. We first note that the function satisfies

$$f(-p) = f(p), \quad \forall\ p \in S^2 \tag{136}$$

and for $\alpha(t) : (-\varepsilon, \varepsilon) \to S^2$ with $\alpha(0) = p \in S^2$, $\alpha'(0) = v \in T_pS^2$, the following holds (note that A is symmetric)

$$\begin{aligned} df_p(v) &= \left.\frac{d}{dt}\right|_{t=0} \langle A\alpha(t),\ \alpha(t)\rangle \\ &= \langle Av,\ p\rangle + \langle Ap,\ v\rangle = 2\langle Ap,\ v\rangle, \quad \forall\ v \in T_pS^2, \quad p \in S. \end{aligned} \tag{137}$$

Therefore, $p \in S^2$ is a critical point of f (same as $df_p = 0$) if and only if $Ap \perp T_pS^2$. Since $p \in S^2$ is also perpendicular to T_pS^2, we have: $p \in S^2$ is a **critical point** of f if and only if

$$Ap = \lambda p \quad \text{for some} \quad \lambda \in \mathbb{R} \ (\lambda = 0 \text{ is possible}), \tag{138}$$

and we have

$$\lambda = \langle \lambda p, \ p \rangle = \langle Ap, \ p \rangle = f(p). \tag{139}$$

Therefore, we conclude the following:

Lemma 2.95 *A point $p \in S^2$ is a **critical point** of $f : S^2 \to \mathbb{R}$ (defined by (135)) if and only if $p \in S^2$ is an **eigenvector** of A with **eigenvalue** $f(p)$. Moreover, if $p \in S^2$ is a **critical point** of f so is $-p \in S^2$ and we have $f(-p) = f(p)$.*

Example 2.96 *Assume the symmetric matrix $A \in M(3)$ has 3 **different** real eigenvalues $\lambda_1, \ \lambda_2, \ \lambda_3$ with*

$$Ap_1 = \lambda_1 p_1, \quad Ap_2 = \lambda_2 p_2, \quad Ap_3 = \lambda_3 p_3, \quad \text{where } p_1, \ p_2, \ p_3 \in S^2.$$

*Then we know that the three vectors $p_1, \ p_2, \ p_3$ must be **independent** in $\mathbb{R}^3$. In such a case, the function $f : S^2 \to \mathbb{R}$ has 6 critical points on S^2, given by $\pm p_1, \ \pm p_2, \ \pm p_3$ and no others. Also, $f : S^2 \to \mathbb{R}$ has 3 critical values on $\mathbb{R}$, given by the above 3 **different** eigenvalues $\lambda_1, \ \lambda_2, \ \lambda_3$ and no others.*

Since S^2 is **compact**, there exist $p_1, \ p_2 \in S^2$ such that $f(p_1) = \min_{S^2} f$ and $f(p_2) = \max_{S^2} f$ and $df_{p_1} = df_{p_2} = 0$. Therefore, we conclude

$$Ap_1 = f(p_1) p_1, \quad Ap_2 = f(p_2) p_2, \quad p_1, \ p_2 \in S^2.$$

Without loss of generality, we can assume $f(p_1) < f(p_2)$ (otherwise, we are in the trivial case $f \equiv c$, a constant, and then $A = cI$). Next, we claim that $p_1 \perp p_2$. To see this, by

$$\begin{aligned} &(f(p_1) - f(p_2)) \langle p_1, \ p_2 \rangle = f(p_1) \langle p_1, \ p_2 \rangle - f(p_2) \langle p_1, \ p_2 \rangle \\ &= \langle Ap_1, \ p_2 \rangle - \langle p_1, \ Ap_2 \rangle = 0, \ \text{where } f(p_1) - f(p_2) \neq 0, \end{aligned}$$

we have $p_1 \perp p_2$. Finally, choose $p_3 \in S^2$ such that $\{p_1, \ p_2, \ p_3\}$ forms an **orthonormal basis**. Then we have

$$\langle Ap_3, \ p_1 \rangle = \langle p_3, \ Ap_1 \rangle = \langle p_3, \ f(p_1) p_1 \rangle = 0$$

and similarly $\langle Ap_3, \ p_2 \rangle = 0$. This implies that $Ap_3 = \lambda p_3$ for some $\lambda \in \mathbb{R}$ and $\lambda = f(p_3)$. We have found an orthonormal basis $\{p_1, \ p_2, \ p_3\}$ with

$$\begin{cases} Ap_1 = f(p_1) p_1 = (\min_{S^2} f \ ) p_1, \\ Ap_2 = f(p_2) p_2 = (\max_{S^2} f) p_2, \\ Ap_3 = f(p_3) p_3, \qquad p_1, \ p_2, \ p_3 \in S^2 \end{cases} \tag{140}$$

which means that A can be **diagonalized**. Finally, we note that

$$\min_{S^2} f \ \leq f(p_3) \leq \max_{S^2} f \tag{141}$$

and it is possible to have $f(p_3) = \min_{S^2} f$ or $f(p_3) = \max_{S^2} f$, but not both.

The above material on regular surface, together with the curve material in Chapter 1, will be the coverage of the midterm on 11/8

<u>To Be Continued after Midterm Exam</u>

See the tex file: **Diff Geom Lecture Note Second Part 2023-11-14.tex**